THE DIOPHANTINE EQUATIONS $x^2 - k = T_n(a^2 \pm 1)$

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It is the object of this note to demonstrate that the two equations of the title have only finitely many solutions in positive integers x and n for any given integers a and k, $k \neq \pm 1$. In these equations, $(T_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind.

1. Chebyshev Polynomials of the First Kind $(T_n(x))_{n\geq 0}$.

These polynomials are defined by the recurrence relation

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x),\ (\forall) x \in C, n \in N^*, \tag{1.1}$$

where $T_0(x) = 1$ and $T_1(x) = x$.

We also have the sequence $(\widetilde{T}_n(x))_{n\geq 0}$ of polynomials "associated" with the Chebyshev polynomials $(T_n(x))_{n\geq 0}$:

$$\widetilde{T}_{n+1}(x) = 2x \cdot \widetilde{T}_n(x) + \widetilde{T}_{n-1}(x),\ x \in C, n \in N^*, \tag{1.2}$$

with $\widetilde{T}_0(x) = 1$ and $\widetilde{T}_1(x) = x$.

The connection between the sequence $(\widetilde{T}_n)_{n\geq 0}$ and the sequence $(T_n)_{n\geq 0}$ is given by the simple relations,

$$\begin{cases} \widetilde{T}_k(x) = \frac{T_k(i\cdot x)}{i^k}, \\ T_k(x) = \frac{\widetilde{T}_k(i\cdot x)}{i^k},\ k \in N, x \in C, \end{cases} \tag{1.3}$$

where $i^2 = -1$.

Two important properties of the polynomials $(T_n)_{n\geq 0}$ are given by the formulas

$$T_n(\cos\varphi) = \cos n\varphi,\ n \in N, \varphi \in C, \tag{1.4}$$

and

$$T_m(T_n(x)) = T_{mn}(x),\ (\forall) m,n \in N, (\forall) x \in C. \tag{1.4'}$$

Also, we observe that

$$\begin{array}{ll}
T_0\left(\frac{x}{\sqrt{2}}\right) = 1 & \widetilde{T}_0\left(\frac{x}{\sqrt{2}}\right) = 1 \\
T_1\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}}\cdot 1 & \widetilde{T}_1\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}}\cdot 1 \\
T_2\left(\frac{x}{\sqrt{2}}\right) = x^2 - 1 & \widetilde{T}_2\left(\frac{x}{\sqrt{2}}\right) = x^2 + 1 \\
T_3\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}}\cdot(2x^2 - 3) & \widetilde{T}_3\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}}\cdot(2x^2 + 3) \\
T_4\left(\frac{x}{\sqrt{2}}\right) = 2x^4 - 4x^2 + 1 & \widetilde{T}_4\left(\frac{x}{\sqrt{2}}\right) = 2x^4 + 4x^2 + 1 \\
T_5\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}}\cdot(4x^4 - 10x^2 + 5) & \widetilde{T}_5\left(\frac{x}{\sqrt{2}}\right) = \frac{x}{\sqrt{2}}\cdot(4x^2 + 10x^2 + 5) \\
\dots & \dots
\end{array}$$

2. The Equation $x^2 - k = T_n(a^2 - 1)$.

Lemma 1: If $(T_n(x))_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind, then one has

$$T_n(a^2 - 1) = 2 \cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) - 1, \ (\forall) n \in N, (\forall) a \in C. \tag{2.1}$$

Proof: Indeed, we have

$$T_n(a^2 - 1) = T_n\left(2 \cdot \left(\tfrac{a}{\sqrt{2}}\right)^2 - 1\right) = T_n\left(T_2\left(\tfrac{a}{\sqrt{2}}\right)\right) = T_{2n}\left(\tfrac{a}{\sqrt{2}}\right)$$
$$= T_2\left(T_n\left(\tfrac{a}{\sqrt{2}}\right)\right) = 2 \cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) - 1. \quad \text{Q.E.D.}$$

Lemma 2: We have

$$2 \cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) = z_m^2, \ z_m \in N^*, \tag{2.2}$$

where $n = 2m+1, \ m \in N$.

Proof: Indeed

$$2 \cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot T_{2m+1}^2\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot \left(\tfrac{a}{\sqrt{2}} \cdot (\cdots)\right)^2$$
$$= a^2 \cdot (\cdots)^2 = (a(\cdots))^2 = z_m^2, \ z_m \in N^*. \quad \text{Q.E.D.}$$

From Lemma 1 and Lemma 2 one obtains, for $n = 2m+1$, $m \in N$, $T_n(a^2 - 1) = T_{2m+1}(a^2 - 1) = z_m^2 - 1$, where $z_m \in Z$. Thus, $x^2 - k = z_m^2 - 1$, which can be solved immediately, giving only finitely many possible values of x, if $k \neq \pm 1$ (see [2]); hence, only finitely many possible corresponding values for $n = 2m+1, m \in N$.

For $n = 2m, m \in N$, from Lemma 1, one obtains

$$T_n(a^2 - 1) + 1 = 2 \cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot z_m^2, \ z_m \in N,$$

where

$$z_m = T_{2m}\left(\tfrac{a}{\sqrt{2}}\right) = T_2\left(T_m\left(\tfrac{a}{\sqrt{2}}\right)\right) = 2 \cdot T_m^2\left(\tfrac{a}{\sqrt{2}}\right) - 1 = 2 \cdot w_m^2 - 1$$

if m is even. If $m = 2\lambda + 1$ is odd, we have

$$z_m = T_{2m}\left(\tfrac{a}{\sqrt{2}}\right) = \begin{cases} v_m^2 - 1, \ m = 2\lambda+1, \lambda \in N, \\ 2w_m^2 - 1, \ m = 2\lambda, \lambda \in N. \end{cases} \tag{2.3}$$

Consequently, one gets

$$x^2 - k = T_n(a^2 - 1) = 2 \cdot T_n^2\left(\tfrac{a}{\sqrt{2}}\right) - 1 = \begin{cases} 2 \cdot (v_m^2 - 1)^2 - 1, & m \text{ odd}, \\ 2 \cdot (2w_m^2 - 1)^2 - 1, & m \text{ even}, \end{cases}$$
$$= \begin{cases} 2 \cdot v_m^4 - 4v_m^2 + 1, & m \text{ odd}, \\ 8w_m^4 - 8w_m^2 + 1, & m \text{ even}. \end{cases} \tag{2.4}$$

Thus, we obtain either

$$x^2 = 2v_m^4 - 4v_m^2 + k + 1 = T_4\left(\tfrac{v_m}{\sqrt{2}}\right) + k \tag{2.5}$$

or

$$x^2 = 8w_m^4 - 8w_m^2 + k + 1 = T_4(w_m) + k, \tag{2.6}$$

and each of these equations has but a finite number of solutions in integers for each given $k = \pm 1$ (see [2]). Thus, for each given $k \in Z, k \neq \pm 1$, there are but finitely many possible values of x, and hence of corresponding $n = 2m, m \in N$.

3. The Equation $x^2 - k = T_n(a^2 + 1)$.

Lemma 3: If $(\widetilde{T}_n)_{n\geq 0}$ is the sequence of polynomials "associated" with the Chebyshev polynomials $(T_n)_{n\geq 0}$, then one has:

(a) $\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^n,\ n \in N;$

(b) $T_n(a^2+1) = \widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right),\ n \in N;$

(c) $T_n(a^2+1) = 2 \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^n,\ n \in N.$

Proof:

(a) We have:

$$\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) = \frac{T_{2n}\left(i \cdot \frac{a}{\sqrt{2}}\right)}{i^{2n}} = (-1)^n \cdot T_{2n}\left(i \cdot \tfrac{a}{\sqrt{2}}\right) = (-1)^n \cdot T_2\left(T_n\left(i \cdot \tfrac{a}{\sqrt{2}}\right)\right)$$

$$= (-1)^n \cdot \left[2 \cdot T_n^2\left(i \cdot \tfrac{a}{\sqrt{2}}\right) - 1\right] = (-1)^n \cdot \left[2 \cdot \left(i^n \cdot \widetilde{T}_n\left(\tfrac{a}{\sqrt{2}}\right)\right)^2 - 1\right]$$

$$= (-1)^n \cdot \left(2 \cdot (-1)^n \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - 1\right) = 2 \cdot \widetilde{T}_n^2\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^n. \quad \text{Q.E.D.}$$

(b)

$$\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) = \frac{T_{2n}\left(i \cdot \frac{a}{\sqrt{2}}\right)}{i^{2n}} = (-1)^n \cdot T_{2n}\left(i \cdot \tfrac{a}{\sqrt{2}}\right) = (-1)^n \cdot T_n\left(T_2\left(i \cdot \tfrac{a}{\sqrt{2}}\right)\right)$$

$$= (-1)^n \cdot T_n\left(2 \cdot \left(\tfrac{ia}{\sqrt{2}}\right)^2 - 1\right) = (-1)^n \cdot T_n(-a^2 - 1)$$

$$= (-1)^n \cdot (-1)^n \cdot T_n(a^2+1). \quad \text{Q.E.D.}$$

(c) For $n = 2m+1, m \in N$, we have

$$\widetilde{T}_{2n}\left(\tfrac{a}{\sqrt{2}}\right) = 2 \cdot \widetilde{T}_{2m+1}^2\left(\tfrac{a}{\sqrt{2}}\right) + 1 = \left(\sqrt{2} \cdot \widetilde{T}_{2m+1}\left(\tfrac{a}{\sqrt{2}}\right)\right)^2 + 1 = z_m^2 + 1,$$

where

$$z_m = \sqrt{2} \cdot \widetilde{T}_{2m+1}\left(\tfrac{a}{\sqrt{2}}\right) \in N^*.$$

Thus, in this case, we obtain $x^2 - k = z_m^2 + 1$, and the result follows as before.

For $n = 2m, m \in N$, we have

$$T_n(a^2+1) = T_{2m}(a^2+1) = 2\cdot \widetilde{T}^2_{2m}\left(\tfrac{a}{\sqrt{2}}\right) - 1 = 2\cdot t_m^2 - 1,$$

where

$$t_m = \widetilde{T}^2_{2m}\left(\tfrac{a}{\sqrt{2}}\right) = 2\cdot \widetilde{T}^2_m\left(\tfrac{a}{\sqrt{2}}\right) - (-1)^m = \begin{cases} v_m^2+1, & m \text{ odd}, \\ 2w_m^2 - 1, & m \text{ even}. \end{cases}$$

Consequently, we have

$$T_n(a^2+1) = T_{2m}(a^2+1) = \begin{cases} 2\cdot(v_m^2+1)^2 - 1, & m \text{ odd}, \\ 2\cdot(2w_m^2-1)^2 - 1, & m \text{ even}, \end{cases} = \begin{cases} 2v_m^4 + 4v_m^2 + 1, & m \text{ odd}, \\ 8w_m^4 - 8w_m^2 + 1, & m \text{ even}. \end{cases} \tag{3.1}$$

Thus, we obtain

$$x^2 = 2v_m^4 + 4v_m^2 + k + 1 = \widetilde{T}_4\left(\tfrac{v_m}{\sqrt{2}}\right) + k \tag{3.2}$$

or

$$x^3 = 8w_m^4 - 8w_m^2 + k + 1 = T_4(w_m) + k \tag{3.3}$$

and the result follows. In this case, as before, for each given $k \neq \pm 1$, there are finitely many possible values of x, and hence, only finitely many possible corresponding values for $n = 2m$, $m \in N$.

This concludes the proof of the result of this paper.

REFERENCES

1. J. H. E. Cohn. "The Diophantine Equation $(x^2-c)^2 = dy^2+4$." *J. London Math. Soc.* **2.8** (1974):253.
2. J. H. E. Cohn. "The Diophantine Equations $(x^2-c)^2 = (t^2 \pm 2)\cdot y^2 + 1$." *Acta Arithmetica* **30** (1976):253-55.

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