SOME IDENTITIES INVOLVING GENERALIZED GENOCCHI POLYNOMIALS AND GENERALIZED FIBONACCI-LUCAS SEQUENCES

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(*Submitted November 1996*)

1. INTRODUCTION

In [7], Toscano gave some novel identities between generalized Fibonacci-Lucas sequences and Bernoulli-Euler polynomials. Later, Zhang and Guo [9] and Wang and Zhang [8] discussed the case of Bernoulli-Euler polynomials of higher order and generalized the results of Toscano.

The purpose of this paper is to establish some identities containing generalized Genocchi polynomials that, as one application, yield some results of Toscano [7] and Byrd [1] as special cases, as well as other identities involving Bernoulli-Euler and Fibonacci-Lucas numbers.

2. SOME LEMMAS

It is well known that a general linear sequence $S_n(p,q)$ $(n=0,1,2,\dots)$ of order 2 is defined by the law of recurrence,

$$S_n(p,q)=pS_{n-1}(p,q)-qS_{n-2}(p,q),$$

with S_0, S_1, p, and q arbitrary, provided that $\Delta=p^2-4q>0$.

In particular, if $S_0=0, S_1=1$, or $S_0=2, S_1=p$, we have generalized Fibonacci and Lucas sequences, respectively, in symbols $U_n(p,q), V_n(p,q)$.

If α,β $(\alpha>\beta)$ are the roots of the equation $x^2-px+q=0$, then we have (see [7])

$$U_{n(}(p,q)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad V_n(p,q)=\alpha^n+\beta^n, \tag{1}$$

$$S_n(p,q)=\left(S_1-\frac{1}{2}S_0\right)U_n(p,q)+\frac{1}{2}S_0V_n(p,q). \tag{2}$$

We assume

$$S_0=k,\quad S_1=\frac{1}{2}pk+\left(x-\frac{1}{2}k\right)\Delta^{1/2}$$

and, using (1) and (2), we deduce that

$$S_n(x;p,q)=\left(x-\frac{1}{2}k\right)\Delta^{1/2}U_n(p,q)+\frac{1}{2}kV_n(p,q), \tag{3}$$

$$S_n(x;p,q)=x\alpha^n+(k-x)\beta^n. \tag{4}$$

From this point on, we shall use the brief notation U_n, V_n, and $S_n(x)$ to denote $U_n(p,q)$, $V_n(p,q)$, and $S_n(x;p,q)$, respectively.

By adapting the method of [7], [8], and [9] to $S_n(x)$, we have obtained the following results:

$$\begin{aligned} &S_n^m(x)+(-1)^\nu S_n^m(k-x) \\ &=\sum_{r=0}^{m}\binom{m}{r}\Delta^{r/2}U_n^r\frac{1}{2^{m-r}}k^{m-r}V_n^{m-r}\left(x-\frac{k}{2}\right)^r(1+(-1)^{\nu+r}) \\ &=\begin{cases}\dfrac{1}{2^{m-1}}\displaystyle\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}k^{m-2r}V_n^{m-2r}(2x-k)^{2r} & (\nu \text{ even}),\\ \dfrac{1}{2^{m-1}}\displaystyle\sum_{r=0}^{[m/2]}\binom{m}{2r+1}\Delta^{r+(1/2)}U_n^{2r+1}k^{m-2r-1}V_n^{m-2r-1}(2x-k)^{2r+1} & (\nu \text{ odd}),\end{cases} \end{aligned} \tag{5}$$

$$\begin{aligned} &S_n^m(x)+(-1)^\nu S_n^m(k-x) \\ &=\sum_{r=0}^{m}\binom{m}{r}q^{nr}[\beta^{n(m-2r)}+(-1)^\nu\alpha^{n(m-2r)}]x^r(k-x)^{m-r} \\ &=\begin{cases}\displaystyle\sum_{r=0}^{m}\binom{m}{r}q^{nr}V_{n(m-2r)}x^r(k-x)^{m-r} & (\nu \text{ even}),\\ -\displaystyle\sum_{r=0}^{m}\binom{m}{r}q^{nr}\Delta^{1/2}U_{n(m-2r)}x^r(k-x)^{m-r} & (\nu \text{ odd}),\end{cases} \end{aligned} \tag{6}$$

$$\begin{aligned} &S_n^m(x)+(-1)^\nu S_n^m(k-x) \\ &=\begin{cases}2\dbinom{2m}{n}q^{nm}x^m(k-x)^m+\displaystyle\sum_{r=0}^{m-1}\binom{2m}{r}q^{nr}V_{2n(m-r)}[x^r(k-x)^{2m-r}+x^{2m-r}(k-x)^r] & (\nu \text{ even}),\\ -\displaystyle\sum_{r=0}^{m-1}\binom{2m}{r}q^{nr}\Delta^{1/2}U_{2n(m-r)}[x^r(k-x)^{2m-r}+x^{2m-r}(k-x)^r] & (\nu \text{ odd}),\end{cases} \end{aligned} \tag{7}$$

$$\begin{aligned} &S_n^{2m+1}(x)+(-1)^\nu S_n^{2m+1}(k-x) \\ &=\begin{cases}\displaystyle\sum_{r=0}^{m}\binom{2m+1}{r}q^{nr}V_{n(2m-2r+1)}[x^r(k-x)^{2m-r+1}+x^{2m-r+1}(k-x)^r] & (\nu \text{ even}),\\ -\displaystyle\sum_{r=0}^{m}\binom{2m+1}{r}q^{nr}\Delta^{1/2}U_{n(2m-2r+1)}[x^r(k-x)^{2m-r+1}+x^{2m-r+1}(k-x)^r] & (\nu \text{ odd}),\end{cases} \end{aligned} \tag{8}$$

and the generating functions

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}U_{nr}=\frac{1}{\Delta^{1/2}}[\exp(t\alpha^n)-\exp(t\beta^n)] \tag{9}$$

and

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}V_{nr}=\exp(t\alpha^n)+\exp(t\beta^n). \tag{10}$$

3. THE MAIN RESULTS

The generalized Genocchi polynomial is defined as (see [4])

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}G_r^{(k)}(x)=\left(\frac{2t}{\exp t+1}\right)^k\exp(tx). \tag{11}$$

From this definition, it is easy to deduce the following properties (see [4]):

$$G_n^{(k)}(k-x)=(-1)^{n+k}G_n^{(k)}(x), \tag{12}$$

$$G_n^{(k+1)}(x)=\frac{2n(k-x)}{k}G_{n-1}^{(k)}(x)+\frac{2}{k}(n-k)G_n^{(k)}(x). \tag{13}$$

In particular, $G_n^{(1)}(0)=G_n$ (Genocchi number, see [3]).

From (11), replacing t by $\Delta^{1/2}U_nt$, we have

$$\begin{aligned}\sum_{r=0}^{\infty}G_r^{(k)}(x)\frac{(\Delta^{1/2}U_nt)^r}{r!}&=\frac{2^k\Delta^{k/2}U_n^kt^k}{[\exp(\Delta^{1/2}U_nt)+1]^k}\exp(x\Delta^{1/2}U_nt)\\&=\frac{2^k\Delta^{k/2}U_n^kt^k}{[\exp(\alpha^nt)+\exp(\beta^nt)]^k}\exp[t(x\alpha^n+(k-x)\beta^n)].\end{aligned}$$

Therefore,

$$[\exp(\alpha^nt)+\exp(\beta^nt)]^k\sum_{r=0}^{\infty}G_r^{(k)}(x)\frac{(\Delta^{1/2}U_nt)^r}{r!}=2^k\Delta^{k/2}U_n^kt^k\exp(tS_n(x)).$$

Hence,

$$\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}V_{nr}\right)^k\left(\sum_{r=0}^{\infty}G_r^{(k)}(x)\frac{(\Delta^{1/2}U_nt)^r}{r!}\right)=2^k\Delta^{k/2}U_n^kt^k\exp(tS_n(x)).$$

We now expand the product figuring in the left member into a power series of t, compare with the expansion of the right member, and obtain

$$\begin{aligned}&\sum_{r=0}^{m}\binom{m}{r}\Delta^{r/2}U_n^rG_r^{(k)}(x)(m-r)!\sum_{r_1+\cdots+r_k=m-r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}\\&=(m)_k2^k\Delta^{k/2}U_n^kS_n^{m-k}(x).\end{aligned} \tag{14}$$

If we replace x by $k-x$ in (14) and use (12), we find

$$\begin{aligned}&\sum_{r=0}^{m}\binom{m}{r}\Delta^{r/2}U_n^r(-1)^rG_r^{(k)}(x)(m-r)!\sum_{r_1+\cdots+r_k=m-r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}\\&=(m)_k(-1)^k2^k\Delta^{k/2}U_n^kS_n^{m-k}(x).\end{aligned} \tag{15}$$

From (14), (15), (5), (7), and (8), we have

$$\begin{aligned}&\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^rU_n^{2r}G_{2r}^{(k)}(x)(m-2r)!\sum_{r_1+\cdots+r_k=m-2r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}\\&=(m)_k2^{k-1}\Delta^{k/2}U_n^k[S_n^{m-k}(x)+(-1)^kS_n^{m-k}(x)]\end{aligned}$$

$$=\begin{cases}\dfrac{(m)_k}{2^{m-2k}}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r}\Delta^r U_n^{2r}k^{m-k-2r}V_n^{m-k-2r}(2x-k)^{2r} & (k\text{ even}),\\ \dfrac{(m)_k}{2^{m-2k}}\Delta^{(k+1)/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r+1}\Delta^r U_n^{2r+1}k^{m-k-2r-1}V_n^{m-k-2r-1}(2x-k)^{2r+1} & (k\text{ odd}),\end{cases}\tag{16}$$

$$=\begin{cases}(m)_k2^{k-1}U_n^k(1+(-1)^{m-k})q^{n(m-k)/2}\dbinom{m-k}{(m-k)/2}x^{\frac{m-k}{2}}(k-x)^{\frac{m-k}{2}}+(m)_k2^{k-1}\Delta^{k/2}U_n^k\\ \quad\times\displaystyle\sum_{r=0}^{m-k-1}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}[x^r(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r] & (k\text{ even}),\\ -(m)_k2^{k-1}\Delta^{(k+1)/2}U_n^k\displaystyle\sum_{r=0}^{\left[\frac{m-k-2}{2}\right]}\binom{m-k}{r}q^{nr}U_{n(m-k-2r)}[x^r(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r] & (k\text{ odd}).\end{cases}\tag{17}$$

We also obtain

$$\sum_{r=0}^{[m/2]}\binom{m}{2r+1}\Delta^{r+(1/2)}U_n^{2r+1}G_{2r+1}^{(k)}(x)(m-2r-1)!\sum_{r_1+\cdots r_k=m-2r-1}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}$$
$$=(m)_k2^{k-1}\Delta^{k/2}U_n^k[S_n^{m-k}(x)-(-1)^kS_n^{m-k}(x)]$$

$$=\begin{cases}\dfrac{(m)_k}{2^{m-2k}}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r+1}\Delta^{r+(1/2)}U_n^{2r+1}k^{m-k-2r-1}V_n^{m-k-2r-1}(2x-k)^{2r+1} & (k\text{ even}),\\ \dfrac{(m)_k}{2^{m-2k}}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r}\Delta^r U_n^{2r}k^{m-k-2r}V_n^{m-k-2r-1}(2x-k)^{2r} & (k\text{ odd}),\end{cases}\tag{18}$$

$$=\begin{cases}(m)_k2^{k-1}\Delta^{k/2}U_n^k(1+(-1)^{m-k})q^{n(m-k)/2}\dbinom{m-k}{(m-k)/2}x^{(m-k)/2}(k-x)^{(m-k)/2}\\ \quad+(m)_k2^{k-1}\Delta^{k/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k-2)/2]}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}[x^r(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r] & (k\text{ odd}),\\ -(m)_k2^{k-1}\Delta^{(k+1)/2}U_n^k\displaystyle\sum_{r=0}^{[(m-k-2)/2]}\binom{m-k}{r}q^{nr}U_{n(m-k-2r)}[x^r(k-x)^{m-k-r}\\ \qquad+x^{m-k-r}(k-x)^r] & (k\text{ even}).\end{cases}\tag{19}$$

4. SOME CONSEQUENCES

If we take $x=k/2$ in (16) and (18), then

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}G_{2r}^{(k)}(k/2)(m-2r)!\sum_{r_1+\cdots+r_k=m-2r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}$$
$$=\begin{cases}\dfrac{(m)_k}{2^{(m-2k)}}\Delta^{k/2}U_n^kk^{m-k}V_n^{m-k} & (k\text{ even}),\\ 0 & (k\text{ odd}),\end{cases}\tag{20}$$

$$\sum_{r=0}^{[m/2]}\binom{m}{2r+1}\Delta^r U_n^{2r} G_{2r+1}^{(k)}(k/2)(m-2r-1)! \sum_{r_1+\cdots+r_k=m-2r-1}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}$$

$$=\begin{cases}0 & (k \text{ even}),\\ \frac{(m)_k}{2^{(m-2k)}}\Delta^{(k-1)/2}U_n^{k-1}k^{m-k}V_n^{m-k} & (k \text{ odd}).\end{cases} \quad (21)$$

Taking $k=2$ in (20), again using $G_{2r}(1)=-G_{2r}$ (see [4]) and recurrence relation (13), we get

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^{r-1}U_n^{2(r-1)}(1-2r)\,G_{2r}\sum_{j}^{m-2r}\binom{m-2r}{j}V_{nj}V_{n(m-2r-j)}=2V_n^{m-2}m(m-1). \quad (22)$$

Taking $k=1$ in (21), using $G_{2r+1}(1/2)=\frac{(2r+1)E_{2r}}{2^{2r}}$, where E_{2r} is the Euler number (see [4]), and $m \to m+1$, we get (22) of [8], namely,

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}\frac{E_{2r}}{2^{2r}}V_{n(m-2r)}=\frac{1}{2^{m-1}}V_n^m. \quad (23)$$

In (22), using $G_{2r}=2(1-2^{2r})B_{2r}$, where B_{2r} is the Bernoulli number (see [2], [4]), we obtain

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^{r-1}U_n^{2(r-1)}(1-2r)(1-2^{2r})B_{2r}\sum_{j}^{m-2r}\binom{m-2r}{j}V_{nj}V_{n(m-2r-j)}=V_n^{m-2}m(m-1). \quad (24)$$

If we take $p=1$ and $q=-1$, then $U_n(1,-1)=F_n$ (*Fibonacci number*), $V_n(1,-1)=L_n$ (*Lucas number*), and from (23) and (24) it follows that

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}5^r F_n^{2r}\frac{E_{2r}}{2^{2r}}L_{n(m-2r)}=\frac{1}{2^{m-1}}L_n^m, \quad (25)$$

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}5^{r-1}F_n^{2r}(1-2r)(1-2^{2r})B_{2r}\sum_{j=0}^{m-2r}\binom{m-2r}{j}L_{nj}L_{n(m-2r-j)}=m(m-1)L_n^{m-2}, \quad (26)$$

where (25) is the result of Byrd [1].

If we take $p=2$ and $q=-1$, then $U_n(2,-1)=P_n$ (*Pell number*), $V_n(2,-1)=Q_n$ (*Pell-Lucas number*, see [5]), and from (23) and (24), it follows that

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}2^r P_n^{2r}E_{2r}Q_{n(m-2r)}=\frac{1}{2^{m-1}}Q_n^m, \quad (27)$$

$$\sum_{r=0}^{[m/2]}\binom{m}{2r}8^{r-1}P_n^{2r}(1-2r)(1-2^{2r})B_{2r}\sum_{j=0}^{m-2r}\binom{m-2r}{j}Q_{nj}Q_{n(m-2r-j)}=m(m-1)Q_n^{m-2}. \quad (28)$$

5. RESULTS IN TERMS OF THE POLYNOMIALS $\Xi_n(u)$, $\mathrm{T}_n(u)$, $\Omega_n(u)$, AND $\Psi_n(u)$

The Bernoulli and Euler polynomials allow themselves to be expressed as follows:

$$B_{2n}(x)=\Xi_n(u),\ u=x^2-x,\ n=0,1,2,\dots,$$
$$E_{2n}(x)=\mathrm{T}_n(u),\ u=x^2-x,\ n=0,1,2,\dots,$$
$$B_{2n-1}(x)=(2x-1)\,\Omega_{n-1}(u),\ u=x^2-x,\ n=1,2,\dots,$$
$$E_{2n-1}(x)=(2x-1)\,\Psi_{n-1}(u),\ u=x^2-x,\ n=1,2,\dots,$$

where Ξ, T, Ω, and Ψ are all polynomials in u (see, e.g., Subramanian and Devanathan [6]). Applying (15), (16), (20), and (21) of [7], we get the following:

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} U_{n(m-2r)} \Xi_r(u) = \frac{m}{2^{m-1}} U_n \sum_{r=0}^{[(m-1)/2]} \binom{m-1}{2r} \Delta^r U_n^{2r} V_n^{m-2r-1} (1+4u)^r, \tag{29}$$

$$\sum_{r=0}^{[(m-2)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} U_{n(m-2r-1)} \Omega_r(u) = \frac{m}{2^{m-1}} U_n \sum_{r=0}^{[(m-2)/2]} \binom{m-1}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-2} (1+4u)^r, \tag{30}$$

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} V_{n(m-2r)} \mathrm{T}_r(u) = \frac{1}{2^{m-1}} U_n \sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} V_n^{m-2r} (1+4u)^r, \tag{31}$$

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} V_{n(m-2r-1)} \Psi_r(u) = \frac{1}{2^{m-1}} U_n \sum_{r=0}^{[(m-1)/2]} \binom{m-1}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-1} (1+4u)^r. \tag{32}$$

6. A REMARK

From our main results (16), (17), (18), and (19), according to different choices of k, x, p, and q, using recurrence relation (13), we can obtain many interesting identities.

Our results have been numerically checked and found to be correct for $m \le 5$.

ACKNOWLEDGMENT

The authors wish to thank the anonymous referee for valuable suggestions and the editor of this journal, Gerald E. Bergum, for posting reference [6], which led to a substantial improvement of the paper.

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AMS Classification Numbers: 11B39, 11B37, 05A15

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