ON CERTAIN SUMS OF FUNCTIONS OF BASE B EXPANSIONS

Curtis Cooper, Robert E. Kennedy, and Milo Renberg
Dept. of Mathematics, Central Missouri State University, Warrensburg, MO 64093-5045
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0. INTRODUCTION

Let $s_b(i)$ denote the base 10 sum of the digits in the base b representation of the nonnegative integer i and $L_b(i)$ denote the number of large digits ($\lceil b/2 \rceil$ or more) in the base b representation of the nonnegative integer i. For example, $s_{10}(4567)=22$, $s_7(7079)=17$ since $7079=26432_7$, and $s_2(19)=3$ since $19=10011_2$. In addition, $L_{10}(4567)=3$, $L_7(7079)=2$, and $L_2(19)=3$. The mathematical literature has many instances of sums involving s_b and L_b. Bush [1] showed that

$$\frac{1}{x}\sum_{n<x} s_b(n) \sim \frac{b-1}{2\log b}\log x.$$

Here, $\log x$ denotes the natural logarithm of x. Mirsky [7], and later Cheo and Yien [2], proved that

$$\frac{1}{x}\sum_{n<x} s_b(n) = \frac{b-1}{2\log b}\log x + O(1).$$

Trollope [9] discovered the following result. Let $g(x)$ be periodic of period one and defined on [0, 1] by

$$g(x)=\begin{cases}\frac{1}{2}x, & 0\le x\le \frac{1}{2},\\ \frac{1}{2}(1-x), & \frac{1}{2}<x\le 1,\end{cases}$$

and let

$$f(x)=\sum_{i=0}^{\infty}\frac{1}{2^i}g(2^i x).$$

Now, if $n=2^m(1+x)$, $0\le x<1$, then

$$\sum_{i<n} s_2(i) = \frac{1}{2\log 2}n\log n - E_2(n),$$

where

$$E_2(n)=2^{m-1}\left\{2f(x)+(1+x)\frac{\log(1+x)}{\log 2}-2x\right\}.$$

In addition, it was shown in [6] that

$$\sum_{i=1}^{\infty}\frac{L_{10}(2^i)}{2^i}=\frac{2}{9}.$$

We will discuss some other sums involving s_b and L_b. In particular, we will give formulas for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m \quad\text{and}\quad \frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m,$$

where m and n are positive integers. Then, we will find a formula for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i).$$

We define $C_b(x; y)$ to be the sum of the carries when the positive integer x is multiplied by y, using the normal multiplication algorithm in base b arithmetic. That is, we convert x and y to base b and then multiply in base b. In this algorithm, we consider the carries above the numbers as well as in the columns. We will prove that

$$\sum_{i=1}^{\infty}\frac{C_b(a; a^i)}{(s_b(a))^i}=\frac{s_b(a)}{b-1}.$$

We will conclude the paper with some open questions.

1. FIRST SUM

To compute

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m,$$

we begin with the function

$$f(x)=(\underbrace{1+\cdots+1}_{\lceil b/2\rceil \text{ times}}+\underbrace{e^x+\cdots+e^x}_{\lfloor b/2\rfloor \text{ times}})^n=(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^n.$$

The motivation for this function comes from the fact that in the base b representation of $i=i_n \ldots i_2 i_1$, the j^{th} digit of i, i_j, is either small or large and thus contributes 0 or 1 to the number of large digits in i. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0\le i\le b^n-1$ and the b^n terms $1\cdot e^{L_b(i)x}$. Therefore,

$$f(x)=(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^n=\sum_{i=0}^{b^n-1}1\cdot e^{L_b(i)x}.$$

Thus,

$$f^{(m)}(x)=\sum_{i=0}^{b^n-1}(L_b(i))^m e^{L_b(i)x},$$

and so we have that

$$f^{(m)}(0)=\sum_{i=0}^{b^n-1}(L_b(i))^m.$$

To continue our discussion, we need the idea of Stirling numbers of the first and second kinds. A discourse on this subject can be found in [3]. A Stirling number of the second kind, denoted by $\{{n \atop k}\}$, symbolizes the number of ways to partition a set of n things into k nonempty subsets. A Stirling number of the first kind, denoted by $[{n \atop k}]$, counts the number of ways to arrange n objects into k cycles. These cycles are cyclic arrangements of the objects. We will use the notation $[A, B, C, D]$ to denote a clockwise arrangement of the four objects A, B, C, and D in a circle. For example, there are eleven different ways to make two cycles from four elements:

$$\begin{array}{llll}[1,2,3][4], & [1,2,4][3], & [1,3,4][2] & [2,3,4][1],\\ [1,3,2][4], & [1,4,2][3], & [1,4,3][2], & [2,4,3][1],\\ [1,2][3,4] & [1,3][2,4], & [1,4][2,3]. & \end{array}$$

Hence, $\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]=11$. Now it can be shown, by induction on m, that

$$f^{(m)}(x)=\sum_{j=1}^{m}\left\{\begin{matrix}m\\j\end{matrix}\right\}n^{\underline{j}}(\lfloor b/2\rfloor e^x)^j(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^{n-j},$$

where $n^{\underline{j}}=n(n-1)\cdots(n-j+1)$. The last quantity is known as the j^{th} falling factorial of n. A discussion of this idea can be found in [3]. Thus,

$$\sum_{i=0}^{b^n-1}(L_b(i))^m=\sum_{j=1}^{m}\left\{\begin{matrix}m\\j\end{matrix}\right\}n^{\underline{j}}\lfloor b/2\rfloor^j\cdot b^{n-j}=b^n\sum_{j=1}^{m}\left\{\begin{matrix}m\\j\end{matrix}\right\}\left(\frac{\lfloor b/2\rfloor}{b}\right)^j n^{\underline{j}}.$$

Since $n^{\underline{j}}=j!\binom{n}{j}$, we have proved the following theorem.

Theorem 1: Let m and n be nonnegative integers. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m=\sum_{j=1}^{m}\left\{\begin{matrix}m\\j\end{matrix}\right\}\left(\frac{\lfloor b/2\rfloor}{b}\right)^j\cdot j!\binom{n}{j}.$$

To illustrate this theorem, if $b=5$, $m=3$, and n is a nonnegative integer, then

$$\frac{1}{5^n}\sum_{i=0}^{5^n-1}(L_5(i))^3=\frac{8}{125}n^3+\frac{36}{125}n^2+\frac{6}{125}n.$$

2. SECOND SUM

Let m and n be positive integers. The determination of the sum

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m$$

was an open question in [4]. In [10], David Zeitlin presented the following answer to the problem in base 10. He stated that if $B_i^{(n)}$ denotes Bernoulli numbers of order n, where

$$\binom{n-1}{i}\cdot B_i^{(n)}=\left[\begin{matrix}n\\n-i\end{matrix}\right],$$

then

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m=\binom{n+m}{m}^{-1}\sum_{i=0}^{m}10^i\cdot\binom{n+m}{m-i}\left\{\begin{matrix}n+i\\n\end{matrix}\right\}\cdot B_{m-i}^{(n)}.$$

To compute

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m,$$

we make use of the function $(g(x))^n$, where $g(x)=1+e^x+e^{2x}+\cdots+e^{(b-1)x}$. The motivation for this function comes from the fact that in the base b representation of $i=i_n\ldots i_2i_1$, the j^{th} digit of i, i_j, contributes i_j to the digital sum of i. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0\le i\le b^n-1$ and the b^n terms $1\cdot e^{s_b(i)x}$. Therefore,

$$(g(x))^n=\sum_{i=0}^{b^n-1}1\cdot e^{s_b(i)x}.$$

Thus, for $m > 1$, we have

$$\frac{d^m}{dx^m}(g(x))^n = \sum_{i=0}^{b^n-1}(s_b(i))^m e^{s_b(i)x},$$

and so we have that

$$\frac{d^m}{dx^m}(g(0))^n = \sum_{i=0}^{b^n-1}(s_b(i))^m.$$

Now we need Faá di Bruno's formula [8]. This formula states that if $f(x)$ and $g(x)$ are functions for which all the necessary derivatives are defined and m is a positive integer, then

$$\frac{d^m}{dx^m}f(g(x)) = \sum_{n_1+2n_2+\cdots+mn_m=m}\frac{m!}{n_1!\ldots n_m!}\left(\frac{d^{n_1+\cdots+n_m}}{dx^{n_1+\cdots+n_m}}f\right)(g(x)) \cdot\left(\frac{\frac{d}{dx}g(x)}{1!}\right)^{n_1}\cdots\left(\frac{\frac{d^m}{dx^m}g(x)}{m!}\right)^{n_m},$$

where $n_1, n_2, \ldots, n_m$ are nonnegative integers.

It follows that

$$\frac{d^m}{dx^m}(g(x))^n = \sum_{n_1+2n_2+\cdots+mn_m=m} n^{\underline{n_1+n_2+\cdots+n_m}} g(x)^{n-n_1-n_2-\cdots-n_m} \cdot\frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)}(x))^{n_1}(g^{(2)}(x))^{n_2}\cdots(g^{(m)}(x))^{n_m},$$

where m is a positive integer and $n_1, n_2, \ldots, n_m$ are nonnegative integers. Thus,

$$\frac{d^m}{dx^m}(g(0))^n = \sum_{n_1+2n_2+\cdots+mn_m=m} n^{\underline{n_1+n_2+\cdots+n_m}} g(0)^{n-n_1-n_2-\cdots-n_m} \cdot\frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)}(0))^{n_1}(g^{(2)}(0))^{n_2}\cdots(g^{(m)}(0))^{n_m}.$$

Equating the two expressions for $\frac{d^m}{dx^m}(g(0))^n$ and simplifying gives the following theorem.

Theorem 2: Let n and m be positive integers and $n_1, n_2, \ldots, n_m$ be nonnegative integers. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m = \sum_{n_1+2n_2+\cdots+mn_m=m}\frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!} \cdot(g^{(1)}(0)/b)^{n_1}(g^{(2)}(0)/b)^{n_2}\cdots(g^{(m)}(0)/b)^{n_m}n^{\underline{n_1+n_2+\cdots+n_m}},$$

where $g^{(i)}(0) = 0^i + 1^i + \cdots + (b-1)^i$.

It might be noted that, in [4], formulas for the sums

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m$$

were given for $m = 0, 1, \ldots, 8$. Using the formulas we just derived, we have the new formula for $m = 9$, that is,

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^9 = \frac{387420489}{512}\cdot n^9 + \frac{1420541793}{128}\cdot n^8 + \frac{12153524229}{256}\cdot n^7 + \frac{7215728751}{160}\cdot n^6 - \frac{30325460319}{512}\cdot n^5 - \frac{2286016425}{128}\cdot n^4 + \frac{30058716303}{640}\cdot n^3 - \frac{2699999973}{160}\cdot n^2.$$

3. THIRD SUM

We next try to tackle the sum

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i).$$

The base 10 result is

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}s_{10}(i)\cdot L_{10}(i) = \frac{9}{4}n^2 + \frac{5}{4}n.$$

From the previous two sections, we have established the formulas

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^2 = \frac{b^2-2b+1}{4}n^2 + \frac{b^2-1}{12}n$$

and

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^2 = \left(\frac{\lfloor b/2\rfloor}{b}\right)^2 n^2 + \left(\left(\frac{\lfloor b/2\rfloor}{b}\right) - \left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n.$$

Now, consider the function

$$h(x) = (1+e^x+e^{2x}+\cdots+e^{(\lceil b/2\rceil-1)x}+e^{(\lceil b/2\rceil+1)x}+\cdots+e^{bx})^n.$$

The motivation for this function comes from the fact that, in the base b representation of $i = i_n \ldots i_2 i_1$, the j^{th} digit of i, i_j, contributes either i_j or i_j+1, depending upon whether or not the i_j^{th} digit is small or large, respectively. That is, the $h(x)$ function considers both the digital sum and the number of large digits, compared to the $g(x)$ function, where we were only concerned with the digital sum. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0 \le i \le b^n-1$ and the b^n terms $1\cdot e^{(s_b(i)+L_b(i))x}$. Therefore,

$$h(x) = (1+e^x+e^{2x}+\cdots+e^{(\lceil b/2\rceil-1)x}+e^{(\lceil b/2\rceil+1)x}+\cdots+e^{bx})^n = \sum_{i=0}^{b^n-1}1\cdot e^{(s_b(i)+L_b(i))x}.$$

Thus,

$$h''(x) = \sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2 e^{(s_b(i)+L_b(i))x},$$

and so we have that

$$h''(0)=\sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2.$$

Computing $h''(0)$ and dividing by b^n, we obtain

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2=n(n-1)b^{-2}\cdot\left(\frac{b(b+1)}{2}-\left\lceil\frac{b}{2}\right\rceil\right)^2+nb^{-1}\cdot\left(\frac{b(b+1)(2b+1)}{6}-\left\lceil\frac{b}{2}\right\rceil^2\right)$$

$$=\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2n^2+\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right)n.$$

But,

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i)=\frac{1}{b^n}\sum_{i=0}^{b^n-1}\frac{(s_b(i)+L_b(i))^2-(s_b(i))^2-(L_b(i))^2}{2}$$

$$=\frac{1}{2}\left(\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i)+L_b(i))^2-\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^2-\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^2\right).$$

Substituting our three formulas in the above expression, we have

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i)=\frac{1}{2}\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2n^2$$

$$+\frac{1}{2}\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right)n$$

$$-\frac{1}{2}\left(\frac{b^2-2b+1}{4}n^2+\frac{b^2-1}{12}n\right)$$

$$-\frac{1}{2}\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)^2n^2+\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n\right).$$

Collecting like terms, we have the following theorem.

Theorem 3: Let n be a positive integer. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(i)\cdot L_b(i)=\frac{1}{2}\left(\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2-\frac{b^2-2b+1}{4}-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n^2$$

$$+\frac{1}{2}\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right.$$

$$\left.-\frac{b^2-1}{12}-\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)\right)n.$$

Furthermore, we have the following corollary.

Corollary: Let n be a positive integer and b be a positive even integer. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i)=\frac{b-1}{4}n^2+\frac{b}{8}n.$$

4. FOURTH SUM

We next determine the sum

$$\sum_{i=1}^{\infty}\frac{C_b(a;a^i)}{(s_b(a))^i}$$

where $C_b(x;y)$ denotes the sum of the carries when the positive integer x is multiplied by y, using the normal multiplication algorithm in base b arithmetic.

Noting that $L_{10}(2^i)=C_{10}(2;2^i)$, this sum is a generalization of the sum

$$\sum_{i=1}^{\infty}\frac{L_{10}(2^i)}{2^i}$$

which was a problem considered in [6].

To compute this sum, we need the following lemma.

Lemma 1: Let d be a digit in base b and y be any positive integer. Then

$$C_b(d;y)=\frac{1}{b-1}(d\cdot s_b(y)-s_b(dy)).$$

Proof: The proof of Lemma 1 relies on Legendre's theorem,

$$s_b(n)=n-(b-1)\sum_{t\geq 1}\left\lfloor\frac{n}{b^t}\right\rfloor,$$

where n is a positive integer. Legendre's theorem and its proof can be found in [5].

To prove Lemma 1, we note that

$$s_b(y)=y-(b-1)\sum_{t\geq 1}\left\lfloor\frac{y}{b^t}\right\rfloor \quad\text{and}\quad s_b(dy)=dy-(b-1)\sum_{t\geq 1}\left\lfloor\frac{dy}{b^t}\right\rfloor.$$

Multiplying the first equality by d and subtracting the second equality from the first yields

$$d\cdot s_b(y)-s_b(dy)=(b-1)\sum_{t\geq 1}\left(\left\lfloor\frac{dy}{b^t}\right\rfloor-d\left\lfloor\frac{y}{b^t}\right\rfloor\right).$$

Dividing by $b-1$ and observing that the sum is $C(d;y)$ gives us the result.

Armed with Lemma 1, we have the next lemma.

Lemma 2: Let $s_b(n)$ denote the base b digital sum of the positive integer n and $C_b(a;a^i)$ denote the base b carries in the normal multiplication algorithm of multiplying a and a^i. Let x and y be positive integers. Then $s_b(x\cdot y)=s_b(x)\cdot s_b(y)-(b-1)C_b(x;y)$.

Proof: Consider $x=\sum_{i=0}^{n}x_ib^i$, the base b representation of x. Then, counting the top carries from the multiplication using Lemma 1 and counting the bottom carries from the addition, we have

$$
\begin{aligned}
C_b(x;y) &= \frac{1}{b-1}\sum_{i=0}^{n}(x_i s_b(y) - s_b(x_i y)) + \sum_{t\ge 1}\left(\left\lfloor\frac{\sum_{i=0}^{n} x_i b^i y}{b^t}\right\rfloor - \sum_{i=0}^{n}\left\lfloor\frac{x_i b^i y}{b^t}\right\rfloor\right)\\
&= \frac{1}{b-1}s_b(x)s_b(y) - \frac{1}{b-1}\sum_{i=0}^{n}s_b(x_i y) + \sum_{t\ge 1}\left\lfloor\frac{xy}{b^t}\right\rfloor - \sum_{i=0}^{n}\sum_{t\ge 1}\left\lfloor\frac{x_i b^i y}{b^t}\right\rfloor\\
&= \frac{1}{b-1}s_b(x)s_b(y) - \frac{1}{b-1}\sum_{i=0}^{n}s_b(x_i y) + \frac{1}{b-1}(xy - s_b(xy))\\
&\quad - \sum_{i=0}^{n}\frac{1}{b-1}(x_i b^i y - s_b(x_i b^i y))\\
&= \frac{1}{b-1}(s_b(x)s_b(y) - s_b(xy)).
\end{aligned}
$$

Next, applying Lemma 2, we obtain $s_b(a^{i+1}) = s_b(a)\cdot s_b(a^i) - (b-1)C_b(a;a^i)$. Thus, if n is a positive integer,

$$
\begin{aligned}
\sum_{i=1}^{n}\frac{C_b(a;a^i)}{s_b(a)^i} &= \frac{1}{b-1}\sum_{i=1}^{n}\left(\frac{s_b(a^i)}{(s_b(a))^{i-1}} - \frac{s_b(a^{i+1})}{(s_b(a))^i}\right)\\
&= \frac{1}{b-1}s_b(a) - \frac{1}{b-1}\frac{s_b(a^{n+1})}{(s_b(a))^n}.
\end{aligned}
$$

Therefore, we have the following theorem.

Theorem 4: Let $s_b(n)$ denote the base b digital sum of the positive integer n and $C_b(a;a^i)$ denote the base b carries in the normal multiplication algorithm of multiplying a and a^i. Then

$$
\sum_{i=1}^{\infty}\frac{C_b(a;a^i)}{(s_b(a))^i} = \frac{s_b(a)}{b-1}.
$$

To illustrate this theorem, if $b=3$ and $a=14$, then

$$
\sum_{i=1}^{\infty}\frac{C_3(14;14^i)}{4^i} = 2.
$$

That is, if we count the carries in multiplying $14 = 112_3$ by powers of 14, using the usual base 3 multiplication algorithm, and divide by the appropriate power of 4, the result is 2. In fact, the infinite series begins with

$$
\frac{5}{4} + \frac{7}{16} + \frac{14}{64} + \frac{18}{256} + \cdots.
$$

5. QUESTIONS

Some open questions remain. Can a formula be found for

$$
\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^{n_1}\cdot(L_b(i))^{n_2},
$$

where n, n_1, and n_2 are positive integers? Can a formula be found for

$$\frac{1}{b^n}\sum_{i=1}^{b^n-1}\frac{1}{s_b(i)}?$$

Also, can a formula be found for

$$\frac{1}{b_1^n}\sum_{i=0}^{b_1^n-1}s_{b_1}(i)\cdot s_{b_2}(i),$$

where $b_1 = b_2^m$? What about a formula for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}s_b(s_b(i))?$$

Finally, find the sum

$$\sum_{i=1}^{\infty}\frac{s_b(a^i)}{a^i}.$$

REFERENCES

1. L. E. Bush. "An Asymptotic Formula for the Average Sum of Digits of Integers." *Amer. Math. Monthly* **47** (1940):154-56.
2. P. Cheo & S. Yien. "A Problem on the *K*-adic Representation of Positive Integers." *Acta Math. Sinica* **5** (1955):433-38.
3. R. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics*, p. 47. Reading, Mass.: Addison-Wesley, 1989.
4. R. E. Kennedy & C. Cooper. "Sums of Powers of Digital Sums." *The Fibonacci Quarterly* **31.4** (1993):341-45.
5. A. M. Legendre. *Théorie des Nombres*, 4th ed., **1**:12. Librairie Scientifique et Technique. Paris: Blanchard, 1955.
6. E. Levine. Problem 386. *College Math. J.* **21** (1990):151-52.
7. L. Mirsky. "A Theorem on Representations of Integers in the Scale of *r*." *Scripta. Math.* **15** (1949):11-12.
8. S. Roman. "The Formula of Faá di Bruno." *Amer. Math. Monthly* **87** (1980):805-09.
9. J. R. Trollope. "An Explicit Expression for Binary Digital Sums." *Math. Mag.* **41** (1968): 21-25.
10. D. Zeitlin. Abstract 94T-11-93: "On a General Exact Formula for Sums of Powers of Digital Sums." *Abstracts of the American Mathematical Society* **95** (1994):431.

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