THE NUMBER OF REPRESENTATIONS OF N USING DISTINCT FIBONACCI NUMBERS, COUNTED BY RECURSIVE FORMULAS

Marjorie Bicknell-Johnson
665 Fairlane Avenue, Santa Clara, CA 95051

Daniel C. Fielder
School of Electrical and Computer Engineering
Georgia Institute of Technology, Atlanta, GA 30332-0250
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0. INTRODUCTION

Let $R(N)$ be the number of representations of the nonnegative integer N as a sum of distinct Fibonacci numbers. For $N = F_n - 1$, $n \geq 1$, the Zeckendorf representation, in which no two consecutive Fibonacci numbers appear in the sum, is the only possible representation, and $R(F_n - 1) = 1$, as proved by Carlitz [3] and Klarner [4]. The sequences $\{b_n - 1\}$, $b_{n+1} = b_n + b_{n-1}$, arise as a generalization, having the property that $R(b_n - 1) = R(b_{n+1} - 1) = k$ for all sufficiently large n (see [1] and [4]). The generation of the specialized and related sequence 1, 3, 8, 16, 24, ..., A_n, whose n^{th} term is the least N such that $n = R(N)$, spurred efforts to find recursive relationships for the values $R(N)$ and ways to compute $R(N)$ for large values of N. Some authors have used $T(N)$ and some $R(N)$ in counting representations; we will use $R(N)$ for the number of ways to represent N as a sum of distinct Fibonacci numbers (without F_1) and $T(N)$ for the number of representations if both F_1 and F_2 are used. In our notation, Carlitz and Klarner both give $R(F_n) = [n/2]$, $n \geq 2$, where $[x]$ is the greatest integer in x. Since $T(N) = R(N) + R(N-1)$, we have concentrated on formulas for $R(N)$.

Earlier authors have used generating functions and combinatorics to develop and prove representation theorems. In this paper we concentrate on properties of the integers whose representations are being counted. We prove Conjectures 1, 2, and 3 from [1] as well as writing formulas for $R(MF_k)$ and $R(ML_k)$, $M \geq 1$, and solving $R(N) = mR(N-1) - q$ for integers M, m, and q.

1. THE SYMMETRIC PROPERTY AND A BASIC RECURSION

The most obvious property in a table of $R(N)$ is the palindromic subsequences it contains, beginning and ending with 1, for N in the interval $F_n - 1 \leq N \leq F_{n+1} - 1$; i.e., when $0 \leq M \leq F_{n-1}$, $n \geq 3$,

$$R(F_{n+1} - 1 - M) = R(F_n - 1 + M). \tag{1}$$

Since these values $R(N)$ are symmetric about the center of each palindromic segment, we only have to compute the values of the first half of the interval. Symmetric property (1) is a variation of Theorem 1, whose results appear in Klarner [5], as specialized for the Fibonacci sequence $\{F_{n+1}\}$.

Theorem 1:

$$R(F_{n+1} - 2 - M) = R(F_n + M),\ 0 \leq M \leq F_{n-1},\ n \geq 3.$$

Values for $R(N)$ for $0 \le N \le 60$

N	$R(N)$	N	$R(N)$	N	$R(N)$	N	$R(N)$
0	1						
1	1	16	4	31	3	46	2
2	1	17	2	32	4	47	5
3	2	18	3	33	1	48	5
4	1	19	3	34	4	49	3
5	2	20	1	35	4	50	6
6	2	21	4	36	3	51	3
7	1	22	3	37	6	52	4
8	3	23	3	38	3	53	4
9	2	24	5	39	5	54	1
10	2	25	2	40	5	55	5
11	3	26	4	41	2	56	4
12	1	27	4	42	6	57	4
13	3	28	2	43	4	58	7
14	3	29	5	44	4	59	3
15	2	30	3	45	6	60	6

It is a simple matter to compute a table for $R(N)$ from generating functions for small N, but as N gets larger, the computer's memory will eventually be exceeded. We have calculated $R(N)$ for $1 \le N \le 257{,}115$ and have capabilities of calculating individual values for $R(N)$ for very large N; for example, $R(3{,}000{,}000{,}000) = 6165$. We have listed $\{A_n\}$ for $1 \le n \le 330$. But to study the mysteries of $\{A_n\}$ or to compute $R(N)$ for large N by hand, we need some recursive relationships. Klarner [5] proved Theorem 2 for generalized Fibonacci numbers.

Theorem 2 (Basic Recursion Formula): If $F_n \le M \le F_{n+1} - 2$, then

$$R(M) = R(F_{n+1} - 2 - M) + R(M - F_n), \; n \ge 4. \tag{2}$$

Lemma 1: If $F_n \le M \le F_{n+1} - 2$, then $R(M - F_n)$ is the number of representations of M using F_n, while the number of representations of M using F_{n-1} is $R(F_{n+1} - 2 - M)$.

Proof: The largest Fibonacci number in M is F_n. $R(M)$ is the sum of the number of representations of M that use F_n and the number of those that use F_{n-1}. Since $M \le F_{n+1} - 2$, no representations of M use both F_n and F_{n-1}; else $M > F_{n+1}$. There are no representations of M that use neither F_n nor F_{n-1}, since $F_n - 2 = F_{n-2} + F_{n-3} + \cdots + F_3 + F_2 < M$. Note that $M = F_n + M_1$, where the largest possible Fibonacci number in M_1 is F_{n-2}; else M could contain F_{n+1}. The number of representations of M that use F_n is $R(M_1) = R(M - F_n)$ since F_n is added to each possible representation of M_1 to make a representation of M using F_n. To list representations of M using F_{n-1}, if we write $M = F_{n-1} + F_{n-2} + M_1$ and then list representations of M_1, there can be a repetition of terms, such as F_{n-2} appearing twice, so we need sums using disjoint sets of Fibonacci numbers. Representations of $(F_{n+1} - 2 - M) = (F_{n-1} + F_{n-2} + \cdots + F_3 + F_2) - M$ will use a set of Fibonacci numbers disjoint from those selected to represent M. Thus, $R(F_{n+1} - 2 - M)$ must give the number of representations of M that use F_{n-1} by examining Theorem 2. □

In counting by hand, $R(M) = R(M - F_n) + R(M - F_{n-1})$ if $M - F_{n-1} < F_{n-1}$. For example, $23 = 21 + 2 = 13 + 10$, and $R(23) = R(2) + R(10)$. If $M - F_{n-1} > F_{n-1}$, an adjustment must be made;

$30 = 21+9 = 13+17 = 13+(13+4)$, and $R(30) = R(9)+R(17)-R(4)$. Lemma 2 makes this counting correction. We take $R(0)=1$ and $R(K)=0$ when $K<0$ in Lemmas 2 through 6, and $[x]$ denotes the greatest integer in x.

Lemma 2: If $F_n \le M \le F_{n+1}-2$, then

$$\begin{aligned} R(M) &= R(M-F_n)+R(M-F_{n-1})-R(M-2F_{n-1}); \\ R(F_{n+1}-2-M) &= R(M-F_{n-1})-R(M-2F_{n-1}). \end{aligned} \tag{3}$$

Proof: $R(M)$ is the number of representations of M using F_n plus the number of representations of M using F_{n-1} corrected for the number of representations of $(M-F_{n-1})$ using F_{n-1}, is any exist. A second way to write the representations of M that use F_{n-1} is to write $M = F_{n-1} + (M-F_{n-1})$ and observe that the number of representations that use F_{n-1} is $R(M-F_{n-1})$ if F_{n-1} is not used in representing $(M-F_{n-1})$. If $M > 2F_{n-1}$, $R(M-2F_{n-1})$ is the number of representations of $(M-F_{n-1})$ using F_{n-1}, since $M-F_{n-1} = F_{n-1}+((M-F_{n-1})-F_{n-1})$. Thus, the representations of M using F_{n-1} are counted by $[R(M-F_{n-1})-R(M-2F_{n-1})]$, which count appeared in Lemma 1 as $R(F_{n+1}-2-M)$. □

Lemma 3:

$$R(F_n+K) = R(F_{n-1}-2-K)+R(K), \quad 0 \le K \le F_{n-1}-2. \tag{4}$$

Lemma 2 is another form of Theorem 2, while Lemma 3 results when $M = K+F_n$ in (2), and is useful in computation. For example, let $K=24$, $R(K)=5$; since $0 \le K \le F_{n-1}-2$, take $n \ge 10$.

$$\begin{aligned} n=12:&\quad R(24+144) = R(87-24)+R(24) = 8+5; \quad R(168)=13, \\ n=13:&\quad R(24+233) = R(142-24)+R(24) = 10+5; \quad R(257)=15, \\ n=14:&\quad R(24+377) = R(231-24)+R(24) = 13+5; \quad R(401)=18, \\ n=16:&\quad R(24+987) = R(608-24)+R(24) = 18+5; \quad R(1011)=23, \end{aligned}$$

where we recognize 24, 168, 257, 401, and 1011 as members of our specialized sequence $\{A_n\}$.

Lemma 4:

$$R(M) = R(M-F_n)+R(M-F_{n-1}), \quad F_n \le M \le F_n+F_{n-3}-1.$$

Proof: Because $2F_{n-1} = F_n+F_{n-3}$, $R(M-2F_{n-1})=0$ in Lemma 2 throughout the interval chosen. □

Lemma 5: $R(N)$ for the interval $F_n \le N \le F_{n+1}-1$ is given by:

$$\begin{aligned} R(F_n+K) &= R(F_{n-2}+K)+R(K), && 0 \le K \le F_{n-3}-1; \\ R(F_n+K) &= 2R(K), && F_{n-3} \le K \le F_{n-2}-1; \\ R(F_n+K) &= R(F_{n+1}-2-K), && F_{n-2} \le K \le F_{n-1}-1. \end{aligned} \tag{5}$$

Proof: Let $M = F_n+K$ in Lemma 4 and use Theorem 1 to write the first and last F_{n-3} values of $R(N)$. Let $F_{n-3}+p = K$ in Lemma 3, followed by application of Theorem 1 since $0 \le p \le F_{n-4}$:

$$\begin{aligned} R(F_n+F_{n-3}+p) &= R(F_{n-1}-2-(F_{n-3}+p))+R(F_{n-3}+p) \\ &= R(F_{n-2}-2-p)+R(F_{n-3}+p) \\ &= R(F_{n-3}+p)+R(F_{n-3}+p). \end{aligned}$$

Thus, $R(F_n+K) = 2R(K)$ when $F_{n-3} \le K \le F_{n-2}-1$. □

Lemma 6:

$$R(F_n+K)=R(F_{n-2}+K)+R(K)-R(K-F_{n-3}),\ 0\le K\le F_{n-1}. \tag{6}$$

Proof: For $0\le K\le F_{n-1}-2$, take $M=F_n+K$ in Lemma 2, so that $M-2F_{n-1}=(M-F_n)+(F_n-2F_{n-1})=K-F_{n-3}$. Then let $K=F_{n-1}-1$ in the expression above, using $R(F_n-1)=1$. Finally, take $K=F_{n-1}$, using $R(F_{n+2})=[(n+2)/2]=R(F_n)+1$ from [3] and [4]. □

2. SPECIAL VALUES FOR $R(b_n-1)$ AND $R(b_n)$

Recursive sequences $\{b_n-1\}$, $b_{n+1}=b_n+b_{n-1}$, have $R(b_n-1)=R(b_{n+1}-1)=k$ for n sufficiently large (see [1] and [4]). We can write sequences for which $R(N-1)=k$, a given constant, as indicated in the following example. Say $k=5$ is given. Find a particular value, i.e., $R(24)=5$. Write $24+1=25=21+3+1$ in Zeckendorf form, or

$$R(24)=R(F_8+F_4+F_1-1)=R(F_8+F_4+F_2-1)=5.$$

These are the first terms, when $F_n=1$, in sequences we seek. Thus,

$$R(F_{n+7}+F_{n+3}+F_n-1)=5=R(F_{n+7}+F_{n+3}+F_{n+1}-1),\ n\ge 1.$$

The symmetric property gives $R(F_{n+7}-1+M)=R(F_{n+8}-1-M)=5$ for $M=F_{n+3}+F_n$, so that we can write

$$R(F_{n+8}-1-(F_{n+3}+F_n))=R(F_{n+7}+F_{n+5}+F_{n+1}-1)=5,\ n\ge 1.$$

Since $R(F_{10})=R(F_{10}+1-1)=5$, again using the symmetric property,

$$R(F_{n+9}+F_n-1)=R(F_{n+9}+F_{n+1}-1)=5,\quad n\ge 1,$$
$$R(F_{n+10}-F_n-1)=R(F_{n+10}-F_{n+1}-1)=5,\quad n\ge 1.$$

Since $R(F_{2k})=R(F_{2k+1})=k$, we can derive in a similar way, for $n\ge 1$:

$$\begin{aligned}&R(F_{2k-1+n}+F_n-1)=k=R(F_{2k-1+n}+F_{n+1}-1);\\&R(F_{2k+n}-F_n-1)=k=R(F_{2k+n}-F_{n+1}-1),\ \text{for } n\ge 1.\end{aligned} \tag{7}$$

For a given value of k, there are many infinite sequences such that $R(b_n-1)=k$. All ways of writing infinite sequences such that $R(b_n-1)=k$, for $k=1,2,3$, were given by Klarner [4] as

$$\begin{aligned}R(F_n-1)&=R(F_{n+1}-1)&&=1;\\R(F_{n+3}+F_n-1)&=R(F_{n+3}+F_{n+1}-1)&&=2;\\R(F_{n+5}+F_n-1)&=R(F_{n+5}+F_{n+1}-1)&&=3;\\R(F_{n+6}-F_n-1)&=R(F_{n+6}-F_{n+1}-1)&&=3.\end{aligned}$$

Some useful equivalent statements are

$$\begin{aligned}R(2F_{n+2}-1)&=R(L_{n+2}-1)&&=2;\\R(3F_{n+3}-1)&=R(4F_{n+3}-1)&&=3;\\R(L_{n+1}+F_n-1)&=R(L_n+F_{n+1}-1)&&=3.\end{aligned}$$

Lemma 7: Let $\{b_n\}$ be a sequence of natural numbers such that $b_{n+2}=b_{n+1}+b_n$. Then $\{b_n\}$ has the following properties:

(i) $R(b_n-1)=R(b_k-1)$ for all $n\ge k$ if F_k is the smallest Fibonacci number used in the Zeckendorf representation of b_k, $k\ge 2$, or if $\{b_n\}$ has $b_2\ge 2b_1$ and $F_{k-1}<b_2-b_1\le F_k$.

(ii) $R(b_n-1)=R(b_n-1)R(F_m)-q$, q a constant, $0\le q\le R(b_n-1)$, where F_m is the smallest Fibonacci number used in the Zeckendorf representation of b_n, $m\ge 2$;

(iii) $R(b_{n+2})=R(b_n)+R(b_n-1)=T(b_n)$, $n\ge k$, as in (i), where $T(N)$ is the number of representations of N as sums of Fibonacci numbers, where both F_1 and F_2 can be used;

(iv) $R(b_{n+2c})=R(b_n)+cR(b_n-1)=R(b_{n+2c-2})+R(b_n-1)$, $n\ge k$.

Proof: Klarner [4] used the Zeckendorf representation of b_n to prove (i) for n sufficiently large; $n\ge k$ as in the second statement appears in [1]. The proof of (ii) relies on Lemma 5 and mathematical induction. Take $F_n\le b_n\le F_{n+1}-1$. Let $b_n=F_n+K$, $0\le K\le F_{n-1}-1$. Assume part (ii) holds for all integers $K=F_{n-1}$. If $0\le K\le F_{n-3}-1$, Lemma 5 and the inductive hypothesis give

$$\begin{aligned}R(b_n)&=R(K)+R(F_{n-2}+K)\\&=[R(K-1)R(F_m)-q_1]+[R(F_{n-2}+K-1)R(F_m)-q_2]\\&=[R(K-1)+R(F_{n-2}+K-1]R(F_m)-(q_1+q_2)\\&=R(F_n+K-1)R(F_m)-q_3\\&=R(b_n-1)R(F_m)-q_3,\quad 0\le q_3<R(b_n-1),\end{aligned}$$

since $0\le q_1+q_2\le R(K-1)+R(F_{n-2}+K-1)=R(F_n+K-1)=R(b_n-1)$, again using the inductive hypothesis. A proof by induction can be made from each of the other two parts of Lemma 5, extending K to the intervals $F_{n-3}\le K\le F_{n-2}-1$, and $F_{n-2}\le K\le F_{n-1}-1$, but is omitted here in the interest of brevity.

To prove (iii), using (i) and (ii),

$$\begin{aligned}R(b_{n+2})&=R(b_{n+2}-1)R(F_{m+2})-q=R(b_n-1)(R(F_m)+1)-q\\&=(R(b_n-1)R(F_m)-q)+R(b_n-1)=R(b_n)+R(b_n-1).\end{aligned}$$

Next, take $N=b_n$ and use $T(N)=R(N)+R(N-1)$ as in [4]. Note: The notation is not standardized; the meanings of $R(N)$ and $T(N)$ are reversed in [4] from those used in this paper. Part (iv) follows from $R(F_{n+2c})=R(F_n)+c$, using (ii) to write

$$\begin{aligned}R(b_{n+2c})&=R(b_{n+2c}-1)R(F_{m+2c})-q=R(b_n-1)(R(F_m)+c)-q\\&=(R(b_n-1)R(F_m)-q)+cR(b_n-1)=R(b_n)+cR(b_n-1),\end{aligned}$$

where, also from (iii) and (i),

$$R(b_{n+2c})=R(b_{n+2c-2})+R(b_{n+2c-2}-1)=R(b_{n+2c-2})+R(b_n-1).\ \square$$

3. FORMULAS FOR $R(N)$ BASED ON ZECKENDORF REPRESENTATION

A formula for $R(N)$ for whole sequences $\{b_n\}$, $b_{n+2}=b_{n+1}+b_n$, can be written, or $R(N)$ for large integers N based on the Zeckendorf representation of N, by repeatedly using Theorem 2, Lemmas 2 and 6, and formulas for $R(F_{n+p}+N)$ as developed next. Let the largest Fibonacci number contained in N be F_n; equivalently, F_n is the largest term in the Zeckendorf representation of N, and $F_n\le N\le F_{n+1}-2$. To count the number of ways to represent N as sums of distinct

Fibonacci numbers, first find the largest two Fibonacci numbers in N and then apply formulas of the form $R(F_{n+p}+N)$.

Lemma 8: Let $F_n \le N \le F_{n+1}-2$. Then

$$\begin{aligned}R(F_{n+1}+N) &= R(N)+R(N-F_n);\\ R(F_{n+2}+N) &= R(N)+R(F_{n+1}-2-N);\\ R(F_{n+3}+N) &= 2R(N).\end{aligned}$$

Proof: Let $M = N+F_{n+1}$ in Lemma 2, where $F_{n+2} \le M < F_{n+3}-2$. Then

$$\begin{aligned}R(F_{n+1}+N) &= R(F_{n+1}+N-F_{n+2})+R(F_{n+1}+N-F_{n+1})-R(F_{n+1}+N-2F_{n+1})\\ &= R(N-F_n)+R(N)-R(N-F_{n+1}) = R(N-F_n)+R(N),\end{aligned}$$

where $R(N-F_{n+1}) = 0$ because $N < F_{n+1}$.

Let $M = N+F_{n+3}$ in Lemma 2, where $F_{n+3} \le M < F_{n+4}-2$;

$$\begin{aligned}R(F_{n+3}+N) &= R(F_{n+3}+N-F_{n+3})+R(F_{n+3}+N-F_{n+2})-R(F_{n+3}+N-2F_{n+2})\\ &= R(N)+R(N+F_{n+1})-R(N-F_n)\\ &= R(N)+[R(N-F_n)+R(N)]-R(N-F_n) = 2R(N).\end{aligned}$$

Let $M = N+F_{n+2}$ in Theorem 2, where $F_{n+2} \le M < F_{n+3}-2$;

$$\begin{aligned}R(F_{n+2}+N) &= R(F_{n+3}-2-(F_{n+2}+N))+R((F_{n+2}+N)-F_{n+2}))\\ &= R(F_{n+1}-2-N)+R(N). \ \square\end{aligned}$$

Theorem 3: Let $F_n \le N \le F_{n+1}-2$. Then

$$R(F_{n+2k+1}+N) = (k+1)R(N),\ k \ge 1; \tag{9}$$

$$R(F_{n+2k}+N) = kR(N)+R(F_{n+1}-2-N),\ k \ge 1. \tag{10}$$

Proof: Assume that $R(F_{n+2j+1}+N) = (j+1)R(N)$ holds for $j \le k$; the case $k=1$ was established in Lemma 8. Consider

$$R(F_{n+2(k+1)+1}+N) = R(F_{(n+2k+1)+2}+N),\ n < F_{n+1} < F_{(n+2k+3)-3}.$$

By the first part of Lemma 5,

$$\begin{aligned}R(F_{n+2k+3}+N) &= R(F_{n+2k+1}+N)+R(N)\\ &= (k+1)R(N)+R(N) \ = \ [(k+1)+1]R(N),\end{aligned}$$

establishing the formula for $R(F_{n+2k+1}+N)$ by induction.

The proof of the even case is similar, again taking the case $k=1$ from Lemma 8, and using Lemma 5; therefore, it is omitted here. $\square$

Theorem 3 can be used as a reduction formula to write $R(N)$ for large N. For example,

$$R(1694) = R(F_{17}+97) = 3R(97)+R(144-2-97) = 3(9)+6 = 33,$$

so $R(1694) = 33$ since $R(97) = 9$ and $R(45) = 6$ are known from data. However, Theorem 3 can be written in another form that is even more useful for computation, as given in Corollary 3.1.

Corollary 3.1: Let $F_n \le N \le F_{n+1} - 2$. Then

$$R(F_m + N) = R(F_{m-n+1})R(N) + r, \quad m - n \ge 2, \tag{11}$$

where $r = 0$ if $m-n$ is odd, and $r = R(F_{n+1} - 2 - N)$ if $m-n$ is even.

Proof: The result follows from $R(F_n) = [n/2]$ for $[x]$ the greatest integer in x from [3] and [4]. Let $m-n=2k+1$, then $m=n+2k+1$ and $[(m-n+1)/2]=k+1$; therefore, $R(F_m+N) = [(m-n+1)/2]R(N)$ by (9). Similarly, let $m-n=2k$ in (10). □

4. SPECIAL VALUES FOR $R(F_n \pm K)$

We write some special formulas useful in breaking down expressions for $R(N)$ by putting special values into equation (1) and Corollary 3.1. Expressions for $k = 0, 1$, and 2 in Lemma 9 appear in [4]. We also find integers m and q such that $R(M) = mR(M-1) - q$.

Lemma 9–Special values for $R(F_n - 1 \pm k)$: Let $[x]$ be the greatest integer contained in x, and let $0 \le k \le F_{n-1}$. Then $R(F_n - 1 + k) = R(F_{n+1} - 1 - k)$ has the following values, $0 \le k \le 8$.

$$\begin{array}{llllll}
k=0: & R(F_n-1) & =R(F_{n+1}-1) & & =1, & n\ge 2; \\
k=1: & R(F_n) & =R(F_{n+1}-2) & =R(F_n) & =[n/2], & n\ge 3; \\
k=2: & R(F_n+1) & =R(F_{n+1}-3) & =R(F_{n-1}) & =[(n-1)/2], & n\ge 4; \\
k=3: & R(F_n+2) & =R(F_{n+1}-4) & =R(F_{n-2}) & =[(n-2)/2], & n\ge 5; \\
k=4: & R(F_n+3) & =R(F_{n+1}-5) & & =n-3, & n\ge 6; \\
k=5: & R(F_n+4) & =R(F_{n+1}-6) & =R(F_{n-3}) & =[(n-3)/2], & n\ge 6; \\
k=6: & R(F_n+5) & =R(F_{n+1}-7) & & =n-4, & n\ge 7; \\
k=7: & R(F_n+6) & =R(F_{n+1}-8) & & =n-4, & n\ge 7; \\
k=8: & R(F_n+7) & =R(F_{n+1}-9) & =R(F_{n-4}) & =[(n-4)/2], & n\ge 7.
\end{array}$$

Lemma 10–Special values for $R(F_{2c} \pm K)$ and $R(F_{2c+1} \pm K)$: Considering n even and n odd, $R(F_n \pm K)$ has the following values:

$$\begin{array}{lll}
R(F_{2c}) & =R(F_{2c+1}) & =R(F_{2c-2})+1; \\
R(F_{2c}+1) & =R(F_{2c-1}) & =R(F_{2c-2}); \\
R(F_{2c+1}+1) & =R(F_{2c}) & =R(F_{2c+1}); \\
R(F_{2c+1}+2) & =R(F_{2c-1}) & =R(F_{2c}+2); \\
R(F_{2c+1}-1) & =R(F_{2c}-1) & =1.
\end{array}$$

Lemma 11: Let K be an integer whose Zeckendorf representation has $F_m + F_k$ for its smallest two terms.

If $k=2$ so that K ends with F_m+1, $m \ge 4$, then
$$R(K) = R(K-1),\ m \text{ odd};\ R(K) = R(K-1) - R(K-2),\ m \text{ even}; \tag{12}$$

If $k=3$ so that K ends in F_m+2, $m \ge 5$, then
$$R(K) = R(K-1) - R(K-3),\ m \text{ odd};\ R(K) = R(K-1),\ m \text{ even}; \tag{13}$$

$$\text{If } K \text{ ends in } F_m + F_{2c},\ 2c \ge 4, \text{ then } R(K) = R(K-1) + R(K+1); \tag{14}$$

$$\text{If } K \text{ ends in } F_m + F_{2c+1},\ 2c \ge 4, \text{ then } R(K) = R(K-1) + R(K+2). \tag{15}$$

Proof: A proof can be written by induction following this outline. Calculate (12) for $K = F_{2c}+1$ and $K = F_{2c+1}+1$. Equation (12) can also be calculated for $K = F_m + F_{2c}+1$ and $K = F_m + F_{2c+1}+1$. Then assume that (12) holds for all K such that $K \le F_{n-1}-1$ and use (5) from Lemma 5, $R(F_n + K) = R(F_{n-2}+K) + R(K)$, $0 \le K \le F_{n-3}-1$, calculating each part of (12). Repeat for the other two parts of Lemma 5. (14) and (15) can be proved by substitution into (12) and (13). When K ends in $F_m + F_{2c}$, $K+1$ ends in $F_m + F_{2c}+1$, so replacing K by $K+1$ in (12) in the even case yields (14). When K ends in $F_m + F_{2c+1}$, then $K+1$ ends in $F_m + F_{2c+1}+1$, which means that $R(K+1) = R(K)$ for the odd case of (12). Also, $K+2$ ends in $F_m + F_{2c+1}+2$, which means that $R(K+2) = R(K+1) - R(K-1)$ from the odd case of (13). Putting these together gives (15). □

Theorem 4: Let $F_m + F_k$ be the smallest Fibonacci numbers in the Zeckendorf representation of M. Then

$$R(M) = R(M-1)R(F_k) - q, \quad 0 \le q < R(M-1). \tag{16}$$

If the Zeckendorf representation of M ends in $F_{2c+1}+1$ or $F_{2c}+2$, where $2c \ge 4$, then $q = 0$. If M ends in $F_{2c+1}+2$, $q = R(M-3)$; $F_{2c}+1$, $q = R(M-2)$. If $m-k$ is odd, $q = 0$. If M ends in $F_{2w}+F_{2c}$, $2c \ge 4$, then $q = (c-1)R(M-1) - R(M+1)$; if M ends in $F_{2w+1}+F_{2c+1}$, $2c \ge 4$, then $q = (c-1)R(M-1) - R(M+2)$.

Proof: Apply Lemma 7(ii) and Lemma 11. When $m-k$ is odd, $q = 0$ by Theorem 3. □

Corollary 4.1: Let K be an integer whose Zeckendorf representation has smallest two terms $F_m + F_k$. Then $R(K) = cR(K-1)$ when $k = 2c$ and m is odd, and when $k = 2c+1$ and m is even.

5. $R(MF_k)$ AND $R(ML_k)$

Below, $R(MF_k)$ can be obtained by putting MF_k into Zeckendorf form and then applying Theorem 3 repeatedly. We list Zeckendorf representations of MF_k for $M \le 18$, taking smallest entry $F_{k-2c} \ge F_2$ and write $R(MF_k)$ for $M \le 29 = L_7$.

$$\begin{array}{rlll}
 & 2F_k &= F_{k+1}+F_{k-2} & \\
L_2F_k = & 3F_k &= F_{k+2}+F_{k-2} & \\
L_3F_k = & 4F_k &= F_{k+2}+F_k+F_{k-2} & = F_{k+3}-F_{k-3} \\
 & 5F_k &= F_{k+3}+F_{k-1}+F_{k-4} & = F_{k+3}+F_k-F_{k-3} \\
 & 6F_k &= F_{k+3}+F_{k+1}+F_{k-4} & \\
L_4F_k = & 7F_k &= F_{k+4}+F_{k-4} & \\
 & 8F_k &= F_{k+4}+F_k+F_{k-4} & = F_{k+4}+F_k+F_{k-4} \\
 & 9F_k &= F_{k+4}+F_{k+1}+F_{k-2}+F_{k-4} & = F_{k+4}+2F_k+F_{k-4} \\
 & 10F_k &= F_{k+4}+F_{k+2}+F_{k-2}+F_{k-4} & = F_{k+4}+3F_k+F_{k-4} \\
L_5F_k = & 11F_k &= F_{k+4}+F_{k+2}+F_k+F_{k-2}+F_{k-4} & = F_{k+4}+4F_k+F_{k-4} = F_{k+5}-F_{k-5} \\
 & 12F_k &= F_{k+5}+F_{k-1}+F_{k-3}+F_{k-6} & = F_{k+5}+F_k-F_{k-5} \\
 & 13F_k &= F_{k+5}+F_{k+1}+F_{k-3}+F_{k-6} & = F_{k+5}+2F_k-F_{k-5} \\
 & 14F_k &= F_{k+5}+F_{k+2}+F_{k-3}+F_{k-6} & = F_{k+5}+3F_k-F_{k-5} \\
 & 15F_k &= F_{k+5}+F_{k+2}+F_k+F_{k-3}+F_{k-6} & = F_{k+5}+4F_k-F_{k-5} \\
 & 16F_k &= F_{k+5}+F_{k+3}+F_{k-1}+F_{k-6} & = F_{k+5}+5F_k-F_{k-5} \\
 & 17F_k &= F_{k+5}+F_{k+3}+F_{k+1}+F_{k-6} & = F_{k+5}+6F_k-F_{k-5} \\
L_6F_k = & 18F_k &= F_{k+6}+F_{k-6} &
\end{array}$$

Lemma 12: For MF_k such that $L_{2c-1} < M \le L_{2c+1}$, $k \ge 2c+2$, the smallest Fibonacci number in the Zeckendorf representation of MF_k is F_{k-2c}, and the largest is F_{k+2c-1} or F_{k+2c}, depending upon the interval, where

$$F_{k+2c-1} \le MF_k < F_{k+2c}, \qquad L_{2c-1} < M < L_{2c};$$
$$F_{k+2c} \le MF_k < F_{k+2c+1}, \qquad L_{2c} \le M \le L_{2c+1}.$$

Proof: Lemma 12 is illustrated for $M \le 18$. Assume it holds for all integers $0 \le Q \le L_{2c-1}$; i.e., the largest term in QF_k is F_{k+2c-2} and the smallest is F_{k-2c-2} when $L_{2c-2} \le Q \le L_{2c-1}$. Since $L_{2c}F_k = F_{k+2c} + F_{k-2c}$ (see [6]), $MF_k = QF_k + L_{2c}F_k = F_{k+2c} + QF_k + F_{k-2c}$ has largest term F_{k+2c} and smallest term F_{k-2c} for $L_{2c} \le M = L_{2c} + Q \le L_{2c+1}$. The subscript difference between F_{k-2c-2} and the next smallest Fibonacci number used in the Zeckendorf representation of MF_k is even. For $L_{2c-1} < M < L_{2c}$, since $L_{2c-1}F_k = F_{k+2c-1} - F_{k-2c+1}$ (see [6]), $MF_k = L_{2c-1}F_k + QF_k = F_{k+2c-1} - F_{k-2c+1} + QF_k$, $0 < Q < L_{2c-2}$.

Assume the largest possible term in the Zeckendorf representation of QF_k is F_{k+2c-3} and the smallest term is F_{k-2i} for $L_{2c-3} < Q < L_{2c-2}$. There is no modification of terms for the Zeckendorf representation in adding F_{k+2c-1}, but the smallest term in the Zeckendorf representation of MF_k becomes F_{k-2c} for $L_{2c-1} < M < L_{2c}$ since

$$\begin{aligned} F_{k-2i} - F_{k-2c+1} &= (F_{k-2i} - F_{k-2c+2}) + F_{k-2c} \\ &= (F_{k-2i-1} + F_{k-2i-3} + \cdots + F_{k-2c+3}) + F_{k-2c}. \end{aligned}$$

Thus, the largest term is $F_{2k+2c-1}$ and the smallest is F_{k-2c} for MF_k, when $L_{2c-1} < M < L_{2c}$. Note that the subscript difference between F_{k-2c} and the next smallest Fibonacci number used in the Zeckendorf representation is odd. □

$R(MF_k)$, $1 \le M \le 29 = L_7$, $k \ge 2c+2$ for Smallest Term F_{k-2c}

$R(F_k)$	$= R(F_{k-0})$		
$R(2F_k)$	$= 2R(F_{k-2})$		
$R(3F_k)$	$= 3R(F_{k-2}) - 1$		
$R(4F_k)$	$= 3R(F_{k-2}) - 2$	$= 3R(F_{k-4}) + 1$	$4 = L_3$
$R(5F_k)$	$= 5R(F_{k-4})$		
$R(6F_k)$	$= 5R(F_{k-4})$		
$R(7F_k)$	$= 5R(F_{k-4}) - 1$		$7 = L_4$
$R(8F_k)$	$= 8R(F_{k-4}) - 3$		
$R(9F_k)$	$= 8R(F_{k-4}) - 4$		
$R(10F_k)$	$= 8R(F_{k-4}) - 5$		
$R(11F_k)$	$= 5R(F_{k-4}) - 4$	$= 5R(F_{k-6}) + 1$	$11 = L_5$
$R(12F_k)$	$= 10R(F_{k-6})$		
$R(13F_k)$	$= 13R(F_{k-6})$		
$R(14F_k)$	$= 12R(F_{k-6})$		
$R(15F_k)$	$= 12R(F_{k-6})$		
$R(16F_k)$	$= 13R(F_{k-6})$		
$R(17F_k)$	$= 10R(F_{k-6})$		
$R(18F_k)$	$= 7R(F_{k-6}) - 1$		$18 = L_6$
$R(19F_k)$	$= 15R(F_{k-6}) - 4$		
$R(20F_k)$	$= 18R(F_{k-6}) - 6$		
$R(21F_k)$	$= 21R(F_{k-6}) - 8$		
$R(22F_k)$	$= 16R(F_{k-6}) - 7$		
$R(23F_k)$	$= 20R(F_{k-6}) - 10$		
$R(24F_k)$	$= 20R(F_{k-6}) - 10$		
$R(25F_k)$	$= 16R(F_{k-6}) - 9$		
$R(26F_k)$	$= 21R(F_{k-6}) - 13$		
$R(27F_k)$	$= 18R(F_{k-6}) - 12$		
$R(28F_k)$	$= 15R(F_{k-6}) - 11$		
$R(29F_k)$	$= 7R(F_{k-6}) - 6$	$= 7R(F_{k-8}) + 1$	$29 = L_7$

Theorem 5: When $L_{2c-1} < M \le L_{2c+1}$, $k \ge 2c+2$,

$$R(MF_k) = R(MF_k - 1)R(F_{k-2c}) - q, \tag{17}$$

where $R(MF_k - 1) = R(MF_{2c+2} - 1)$. Further, $q = 0$ for $L_{2c-1} < M < L_{2c}$ while $q = R(MF_{2c+2} - 2)$ for $L_{2c} \le M \le L_{2c+1}$.

Proof: The assertions follow from Theorem 4 by taking $k = 2c+2$ in (17), since we have F_{k-2c} as the smallest term of the Zeckendorf representation of MF_k by Lemma 12. When $L_{2c-1} < M < L_{2c}$, the last two terms in the Zeckendorf representation are $F_m + F_{k-2c}$, where $(m-k+2c)$ is odd; thus, in using Theorem 3 repeatedly to evaluate $R(MF_k)$ from its Zeckendorf representation, we will have $q = 0$ by Corollary 3.1. When $L_{2c} \le M \le L_{2c+1}$, the subscripts of the last two terms will have an even difference, so a remainder term will be involved. Taking $k = 2c+2$ to give the smallest $F_m = F_2$ gives $q = R(MF_{2c+2} - 2)$ by Theorem 4 in the interval where $q \ne 0$. □

Next, we note that the values $R(MF_{2c+2} - 1)$ form palindromic subsequences such that:

$$R((L_{2c-1} + K)F_k - 1) = R((L_{2c} - K)F_k - 1), \quad 1 \le K \le [L_{2c-2}/2];$$
$$R((L_{2c} + K)F_k - 1)) = R((L_{2c+1} - K)F_k - 1), \quad 0 \le K \le [L_{2c-1}/2].$$

Also of interest, we have

$$R(L_{2c}F_k - 1) = R(L_{2c+1}F_k - 1);$$
$$R(L_{2c-1}F_k - 1) + 2 = R(L_{2c}F_k - 1).$$

Corollary 5.1: $R(L_nL_p - 1) = 4(p-1)$, $n \ge p+3$, $p \ge 2$.

Proof: Vajda [6] gives equation (17a), equivalent to

$$\begin{cases} L_{n+p} + L_{n-p} = L_nL_p, & p \text{ even}, \\ L_{n+p} - L_{n-p} = L_nL_p, & p \text{ odd}. \end{cases}$$

Since $L_{n+p} + L_{n-p} = F_{n+p+1} + F_{n+p-1} + F_{n-p+1} + F_{n-p-1}$, the smallest Fibonacci number used in the Zeckendorf representation is F_{n-p-1}. Theorem 4 gives

$$\begin{aligned} R(L_{n+p} + L_{n-p}) &= R(L_{n+p} + L_{n-p} - 1)R(F_{n-p-1}) - q \\ &= R(L_nL_p - 1)R(F_{n-p-1}) - q. \end{aligned}$$

Since we only want $R(L_nL_p - 1)$, we calculate $R(L_{n+p} + L_{n-p} - 1)$ when $F_{n-p-1} = F_2$ or, for $n-p = 3$, $n = p+3$, so that $R(L_{n+p} + L_{n-p} - 1)$ has a constant value for $n \ge p+3$.

$$\begin{aligned} R(L_{n+p} + L_{n-p} - 1) &= R(L_{2p+3} + L_3 - 1) \\ &= R(F_{2p+4} + F_{2p+2} + 3) \\ &= R(F_{2p+2} + 3) + R(F_{2p+1} - 5) \\ &= (2p-1) + (2p-3) = 4(p-1), \end{aligned}$$

where we have applied earlier formulas from Theorem 3 and special values for $R(F_{n+1} - 1 - K)$. Thus, $R(L_nL_p - 1) = 4(p-1)$ for p even.

Similarly, for p odd, $L_{n+p} - L_{n-p}$ has F_{n-p-2} as the smallest Fibonacci number in its Zeckendorf representation. Again calculate $R(L_{n+p} - L_{n-p} - 1)$ for the smallest value for $F_{n-p-2} = F_2$, which occurs for $n-p=4$, $n = p+4$. Then

$$\begin{aligned} R(L_{2p+4} - L_4 - 1) &= R(F_{2p+5} + F_{2p+3} - 8) \\ &= 2R(F_{2p+3} - 8) = 2(2p-2) = 4(p-1). \end{aligned}$$

Thus, $R(L_nL_p - 1) = 4(p-1)$ for p odd, establishing Corollary 5.1 and proving Conjecture 2 of [1]. □

Corollary 5.2: $R(F_pF_n - 1) = F_p$, $n \ge p$, $p \ge 3$.

Proof: $F_{2c+1}F_k$ and $F_{2c+2}F_k$ both have F_{k-2c} as the smallest term in the Zeckendorf representation. Thus,

$$R(F_{2c+1}F_k) = R(F_{2c+1}F_k - 1)R(F_{k-2c}) - q;$$
$$R(F_{2c+2}F_k) = R(F_{2c+2}F_k - 1)R(F_{k-2c}) - q.$$

When $k \ge 2c+2$, $R(MF_k - 1)$ has a constant value. When $k = 2c+2$,

$$R(F_{2c+1}F_k - 1) = R(F_{2c+1}F_{2c+2} - 1) = F_{2c+1}$$

while

$$R(F_{2c+2}F_k - 1) = R(F_{2c+2}F_{2c+2} - 1) = 2c+2,$$

applying two identities from Carlitz [3]. Thus, $R(F_pF_n - 1) = F_p$, establishing Corollary 5.2 and making a second proof of Theorem 3 in [1]. □

The Lucas case $R(ML_k)$ is very similar, relying on [6] for $F_pL_k = F_{k+p} + F_{k-p}$, p odd, and $F_pL_k = F_{k+p} - F_{k-p}$, p even. When $F_{2c-2} < M \le F_{2c}$, the smallest term in the Zeckendorf representation of ML_k is F_{k-2c+1}, while the largest is F_{k+2c-2}, $F_{2c-2} < M < F_{2c-1}$, or F_{k+2c-1}, $F_{2c-1} \le M \le F_{2c}$, $k \ge 2c+1$.

Zeckendorf Representations for ML_k, $1 \le M \le 13$

$$\begin{array}{rrll}
F_2L_k = & L_k &= F_{k+1} + F_{k-1} & = F_{k+2} - F_{k-2} \\
F_3L_k = & 2L_k &= F_{k+3} + F_{k-3} & \\
F_4L_k = & 3L_k &= F_{k+3} + F_{k+1} + F_{k-1} + F_{k-3} & = F_{k+4} - F_{k-4} \\
& 4L_k &= F_{k+4} + F_{k+1} + F_{k-2} + F_{k-5} & \\
F_5L_k = & 5L_k &= F_{k+5} + F_{k-5} & \\
& 6L_k &= F_{k+5} + F_{k+1} + F_{k-1} + F_{k-5} & \\
& 7L_k &= F_{k+5} + F_{k+3} + F_{k-3} + F_{k-5} & \\
F_6L_k = & 8L_k &= F_{k+5} + F_{k+3} + F_{k+1} + F_{k-1} + F_{k-3} + F_{k-5} & = F_{k+6} - F_{k-6} \\
& 9L_k &= F_{k+6} + F_{k+1} + F_{k-2} + F_{k-4} + F_{k-7} & \\
& 10L_k &= F_{k+6} + F_{k+3} + F_{k-4} + F_{k-7} & \\
& 11L_k &= F_{k+6} + F_{k+3} + F_{k+1} + F_{k-1} + F_{k-4} + F_{k-7} & \\
& 12L_k &= F_{k+6} + F_{k+4} + F_{k+1} + F_{k-2} + F_{k-7} & \\
F_7L_k = & 13F_k &= F_{k+7} + F_{k-7} &
\end{array}$$

$R(ML_k)$, $1 \le M \le 21 = F_8$, $k \ge 2c+1$ for Smallest Term F_{k-2c-1}

$R(L_k) = 2R(F_{k-1}) - 1$	$R(12L_k) = 18R(F_k - 7)$
$R(2L_k) = 4R(F_{k-3}) - 1$	$R(13L_k) = 8R(F_{k-7}) - 1$
$R(3L_k) = 4R(F_{k-3}) - 3$	$R(14L_k) = 24R(F_{k-7}) - 7$
$R(4L_k) = 8R(F_{k-5})$	$R(15L_k) = 30R(F_{k-7}) - 11$
$R(5L_k) = 6R(F_{k-5}) - 1$	$R(16L_k) = 20R(F_{k-7}) - 9$
$R(6L_k) = 12R(F_{k-5}) - 5$	$R(17L_k) = 32R(F_{k-7}) - 16$
$R(7L_k) = 12R(F_{k-5}) - 7$	$R(18L_k) = 20R(F_{k-7}) - 11$
$R(8L_k) = 6R(F_{k-5}) - 5$	$R(19L_k) = 30R(F_{k-7}) - 19$
$R(9L_k) = 18R(F_{k-7})$	$R(20L_k) = 24R(F_{k-7}) - 17$
$R(10L_k) = 16R(F_{k-7})$	$R(21L_k) = 8R(F_{k-7}) - 7$
$R(11L_k) = 16R(F_{k-7})$	

Theorem 6: When $F_{2c-2} < M \le F_{2c}$, $k \ge 2c+1$,

$$R(ML_k) = R(ML_k - 1)R(F_{k-2c+1}) - q, \tag{18}$$

where $R(ML_k - 1) = R(ML_{2c+1} - 1)$; further, $q = 0$ for $F_{2c-2} < M < F_{2c-1}$, and $q = R(ML_{2c+1} - 2)$ when $F_{2c-1} \le M \le F_{2c}$.

The proof of Theorem 6 depends on Theorem 4, and being similar to the proof of Theorem 5 is omitted here. We note that the values $R(ML_{2c+1} - 1)$ form palindromic subsequences such that

$$R((F_{2c-1} + K)L_k - 1) = R((F_{2c} - K)L_k - 1), \quad 0 \le K \le [F_{2c-2}/2];$$
$$R((F_{2c} + K)L_k - 1) = R((F_{2c} - K)L_k - 1), \quad 1 \le K \le [F_{2c-1}/2].$$

6. $R(F_m \pm F_k)$

Theorem 7: $R(F_m \pm F_k)$ and $R(F_m \pm F_k - 1)$ have the following values:

$$\begin{aligned}
R(F_m + F_k) &= R(F_{m-k+2})R(F_k), && (m-k) \text{ odd;}\\
R(F_m + F_k) &= R(F_{m-k+2})R(F_k) - 1, && (m-k) \text{ even;}\\
R(F_m - F_k) &= R(F_{m-k+1})R(F_{k-1}) + 1, && (m-k) \text{ even;}\\
R(F_m - F_k) &= R(F_{m-k+1})R(F_{k-1}), && (m-k) \text{ odd;}\\
R(F_m + F_k - 1) &= R(F_{m-k+2});\\
R(F_m - F_k - 1) &= R(F_{m-k+1}).
\end{aligned}$$

Proof: By Corollary 3.1,

$$R(F_m + F_k) = R(F_{m-k+1})R(F_k) + r.$$

If $m-k$ is odd, $r = 0$, and $R(F_{m-k+1}) = R(F_{m-k+2})$, making $R(F_m + F_k) = R(F_{m-k+2})R(F_k)$. If $m-k$ is even, $r = R(F_{k+1} - 2 - F_k) = R(F_{k-2}) = R(F_k) - 1$, and $R(F_{m-k+1}) + 1 = R(F_{m-k+2})$, making $R(F_m + F_k) = R(F_{m-k+2})R(F_k) - 1$.

Equations (7) give $R(F_m \pm F_k - 1)$ by examining the difference of the subscripts; note that the results for $R(F_m + F_k)$ agree with Theorem 4. Using Theorem 1, followed by Corollary 3.1, while noting that the greatest Fibonacci number in $F_k - 2$ is F_{k-1},

$$R(F_m - F_k) = R(F_{m-1} + (F_k - 2)) = R(F_{m-k+1})R(F_k - 2) + r.$$

Note that $R(F_k-2)=R(F_{k-1})$. If $m-k$ is odd, $r=0$, while if $m-k$ is even,

$$r=R(F_k-2-(F_k-2))=R(0)=1. \ \square$$

Corollary 7.1: $R(F_rL_t-1)$ can be written as

(i) $R(F_nL_p-1)=2R(F_p)+1,\ n\ge p+2,\ p\ge 1;$

(ii) $R(L_nF_p-1)=2R(F_{p+1}),\ n\ge p+1,\ p\ge 2.$

Proof: Vajda [6] gives equation (15a), equivalent to

$$\begin{cases} F_{n+p}+F_{n-p}=F_nL_p, & p \text{ even},\\ F_{n+p}-F_{n-p}=F_nL_p, & p \text{ odd}, \end{cases}$$

By Theorem 7, $R(F_{n+p}+F_{n-p}-1)=R(F_{2p+2})=p+1$, while $R(F_{n+p}-F_{n-p}-1)=R(F_{2p})=p$. So $R(F_nL_p-1)=p+1$, p even, and $R(F_nL_p-1)=p$, p odd, which makes $R(F_nL_p-1)=2[p/2]+1$, proving part (i) as well as Conjecture 3 of [1]. Since [6] also gives

$$\begin{cases} F_{n+p}+F_{n-p}=L_nF_p, & p \text{ odd},\\ F_{n+p}-F_{n-p}=L_nF_p, & p \text{ even}. \end{cases}$$

in the same way, we can show that $R(L_nF_p-1)=p+1$, p odd, and $R(L_nF_p-1)=p$, p even, which can be rewritten in the form of (ii). Thus, we have proved part (ii) as well as Conjecture 1 of [1]. $\square$

Corollary 7.2: Let $F_n\le N<F_{n+1}-2$.

(i) $R(L_{p+1})=2R(F_p)-1=R(L_{p-1})+2,\ p\ge 4;$

(ii) $R(L_{n+p}+N)=R(F_{n+p-1}+N)+R(F_{n+p-3}+N)=R(L_{p+1})R(N)+2r,$
where $r=0$ if p is odd, and $r=R(F_{n+1}-2-N)$ if p is even;

(iii) $R(L_{n+p}-K)=2R(F_{n+p-2}+(K-2)),\ 2\le K\le F_{n+p-3}$.

Proof: Since $L_{p+1}=F_{p+2}+F_p$, let $m=p+2$ and $k=p$ in Theorem 7 to write (i). Apply equation (10) to $R(F_{n+p+1}+F_{n+p-1}+N)$ followed by Theorem 1 to write the first part of (ii). Then use Corollary 3.1 and (i) to simplify, finally obtaining (ii).

When $2\le K\le F_{n+p-3}$, the largest term in the Zeckendorf representation of $F_{n+p-1}-K$ is F_{n+p-2}. Then

$$\begin{aligned} R(L_{n+p}-K)&=R(F_{n+p+1}+(F_{n+p-1}-K))\\ &=2R(F_{n+p-1}-K)=2R(F_{n+p-2}-2+K). \ \square \end{aligned}$$

Corollary 7.3:

$$R(L_{n+p}+L_{n-p})=(2p-2)R(L_{n-p})-1=4(p-1)R(F_{n-p-1})-(2p-1);$$
$$R(L_{n+p}-L_{n-p})=4(p-1)R(F_{n-p-2}),\ n-p\ge 3.$$

Proof: Let $N=L_{n-p}=F_{n-p+1}+F_{n-p-1}$ in Corollary 7.2. Then

$$\begin{aligned} &R(F_{n+p-1}+N)+R(F_{n+p-3}+N)\\ &=(p-1)R(L_{n-p})+(p-2)R(L_{n-p})+2R(F_{n-p+2}-2-F_{n-p+1}-F_{n-p-1}) \end{aligned}$$

$$= (2p-3)R(L_{n-p}) + 2R(F_{n-p-3}) = (2p-3)R(L_{n-p}) + R(L_{n-p}) - 1$$
$$= (2p-2)R(L_{n-p}) - 1 = (2p-2)[2R(F_{n-p-1}) - 1] - 1$$
$$= 4(p-1)R(F_{n-p-1}) - (2p-1).$$

Now let $K = L_{n-p}$ in Corollary 7.2. Then

$$R(L_{n+p} - L_{n-p}) = 2R(F_{n+p-2} + F_{n-p+1} + F_{n-p-1} - 2)$$
$$= 2(p-1)R(F_{n-p+1} + F_{n-p-1} - 2)$$
$$= 2(p-1)(2R(F_{n-p-1} - 2)) = 4(p-1)R(F_{n-p-2}),$$

finishing Corollary 7.3. □

Corollary 7.4: $R(L_nL_p - 1) = 4(p-1),\ n \ge p+3,\ p \ge 2.$

Proof: Vajda [6] gives $L_{n+p} + L_{n-p} = L_nL_p$ when p is even, and $L_{n+p} - L_{n-p} = L_nL_p$ when p is odd. The smallest Fibonacci numbers in the Zeckendorf representations are F_{n-p-1} and F_{n-p-2}, respectively. Since also $R(L_{n+p} \pm L_{n-p} - 1) = R(L_nL_p - 1)$, apply Theorem 4 to Corollary 7.3. This also proves Conjecture 2 in [1]. □

Corollary 7.5: $R(5F_nF_p - 1) = 4(p-1),\ n \ge p+3,\ p \ge 2.$

Proof: $L_{n+p} + L_{n-p} = 5F_nF_p$, p odd; $L_{n+p} - L_{n-p} = 5F_nF_p$, p even, also appear in [6], giving an easy identity as in Corollary 7.4. □

REFERENCES

1. Marjorie Bicknell-Johnson. "A Note on a Representation Conjecture by Hoggatt." In *Applications of Fibonacci Numbers* 7 (to be published by Kluwer Academic Publishers, 1998).
2. Marjorie Bicknell-Johnson & David A. Englund. "Greatest Integer Identities for Generalized Fibonacci Sequences $\{H_n\}$, $H_n = H_{n-1} + H_{n-2}$." *The Fibonacci Quarterly* **33.1** (1995):50-58.
3. L. Carlitz. "Fibonacci Representations." *The Fibonacci Quarterly* **6.4** (1968):193-220.
4. David A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* **6.4** (1968):235-44.
5. David A. Klarner. "Representations of N as a Sum of Distinct Elements from Special Sequences." *The Fibonacci Quarterly* **4.4** (1966):289-306.
6. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester, England: Ellis Horwood Ltd., 1989.

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