ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-550 ***Proposed by Paul S. Bruckman, Highwood, IL***

Suppose n is an odd integer, p an odd prime $\neq 5$. Prove that $L_n \equiv 1 \pmod{p}$ if and only if either (i) $\alpha^n \equiv \alpha$, $\beta^n \equiv \beta \pmod{p}$, or (ii) $\alpha^n \equiv \beta$, $\beta^n \equiv \alpha \pmod{p}$.

H-551 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let k be a nonnegative integer and define the following restricted double-sum,

$$S_k := \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} (br+as)^k,$$

where a, b are relatively prime positive integers.

a. Show that

$$S_{k-1} = \frac{1}{kb}\left[\sum_{r=0}^{b-1}((ab+r)^k - a^k r^k) - \sum_{m=2}^{k}\binom{k}{m} b^m S_{k-m}\right]$$

for $k \geq 1$. The convention that $\binom{k}{m} = 0$ if $m > k$ is adopted.

b. Show that

$$S_2 = \frac{ab}{12}[3a^2b^2 + 2a^2b + 2ab^2 - a^2 - b^2 - 9ab + a + b + 2].$$

H-552 ***Proposed by Paul S. Bruckman, Highwood, IL***

Given $m \geq 2$, let $\{U_n\}_{n=0}^{\infty}$ denote a sequence of the following form:

$$U_n = \sum_{i=1}^{m} a_i(\theta_i)^n,$$

where the a_i's and θ_i's are constants, with the θ_i's distinct, and the U_n's satisfy the initial conditions: $U_n = 0, n = 0, 1, \ldots, m-2$; $U_{m-1} = 1$.

Part A. Prove the following formula for the U_n's:

$$U_n = \sum_{S(n-m+1,\, m)} (\theta_1)^{i_1}(\theta_2)^{i_2} \ldots (\theta_m)^{i_m}, \tag{a}$$

where

$$S(N,m)=\{(i_1,i_2,\dots,i_m): i_1+i_2+\cdots+i_m=N,\ 0\le i_j\le N,\ j=1,2,\dots,m\}. \tag{b}$$

Part B. Prove the following determinant formula for the U_n's:

$$U_n=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^2 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^{n} & (\theta_2)^{n} & (\theta_3)^{n} & \cdots & (\theta_m)^{n} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^2 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^{m-1} & (\theta_2)^{m-1} & (\theta_3)^{m-1} & \cdots & (\theta_m)^{m-1} \end{vmatrix}.$$

SOLUTIONS

Sum Problem

H-535 ***Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy***
(Vol. 35, no. 4, November 1997)

For given positive integers n and m, find a closed form expression for $\sum_{k=1}^n k^m F_k$.

Conjecture by the proposers:

$$\Sigma_{m,n}=\sum_{k=1}^{n} k^m F_k = p_1^{(m)}(n)F_{n+1}+p_2^{(m)}(n)F_n+C_m, \tag{1}$$

where $p_1^{(m)}(n)$ and $p_2^{(m)}(n)$ are polynomials in n of degree m,

$$p_1^{(m)}(n)=\sum_{i=0}^{m}(-1)^i a_{m-i}^{(m)} n^{m-i},\quad p_2^{(m)}(n)=\sum_{i=0}^{m}(-1)^i b_{m-i}^{(m)} n^{m-i}, \tag{2}$$

the coefficients $a_k^{(m)}$ and $b_k^{(m)}$ $(k=0,1,\dots,m)$ are positive integers and C_m is an integer.

On the basis of the well-known identity

$$\Sigma_{1,n}=(n-2)F_{n+1}+(n-1)F_n+2, \tag{3}$$

which is an alternate form of Hoggatt's identity I_{40}, the above quantities can be found recursively by means of the following algorithm:

1. $$p_1^{(m+1)}(n)=(m+1)\int p_1^{(m)}(n)\,dn+(-1)^{m+1}a_0^{(m+1)},$$ $$p_2^{(m+1)}(n)=(m+1)\int p_2^{(m)}(n)\,dn+(-1)^{m+1}b_0^{(m+1)}.$$

2. $$a_0^{(m+1)}=\sum_{i=1}^{m+1}\left(a_i^{(m+1)}+b_i^{(m+1)}\right).$$

3. $$b_0^{(m+1)}=\sum_{i=1}^{m+1}a_i^{(m+1)}.$$

4. $$C_{m+1}=(-1)^m a_0^{(m+1)}.$$

Example: The following results were obtained using the above algorithm.

$$\Sigma_{2,n} = (n^2-4n+8)F_{n+1}+(n^2-2n+5)F_n-8,$$

$$\Sigma_{3,n} = (n^3-6n^2+24n-50)F_{n+1}+(n^3-3n^2+15n-31)F_n+50,$$

$$\Sigma_{4,n} = (n^4-8n^3+48n^2-200n+416)F_{n+1}+(n^4-4n^3+30n^2-124n+257)F_n-416,$$

$$\Sigma_{5,n} = (n^5-10n^4+80n^3-500n^2+2080n-4322)F_{n+1} + (n^5-5n^4+50n^3-310n^2+1285n-2671)F_n+4322.$$

Remarks:

(i) Obviously, these results can be proved by induction on n.

(ii) It can be noted that, using the same algorithm, $\Sigma_{1,n}$ can be obtained by the identity

$$\Sigma_{0,n} = F_{n+1}+F_n-1.$$

(iii) It appears that

$$a_k^{(m+k)}/b_k^{(m+k)} = \text{const.} = a_0^{(m)}/b_0^{(m)} \quad (k=1,2,\dots)$$

and

$$\lim_{m\to\infty} a_0^{(m)}/b_0^{(m)} = \alpha.$$

Solution by Paul S. Bruckman, Highwood, IL

We begin by defining certain polynomials of degree n, as follows:

$$F_{m,n}(x) = \sum_{k=1}^{n} k^m x^k. \tag{1}$$

We then see that

$$\Sigma_{m,n} = 5^{-1/2}\{F_{m,n}(\alpha)-F_{m,n}(\beta)\}. \tag{2}$$

Now let U denote the operator xd/dx. It is easily seen that

$$U(F_{m,n}(x)) = F_{m+1,n}(x). \tag{3}$$

Note that

$$F_{0,n}(x) = (x^{n+1}-x)/(x-1). \tag{4}$$

Repeated application of the recurrence in (3) yields the following:

$$F_{1,n}(x) = \{nx^{n+2}-(n+1)x^{n+1}+x\}/(x-1)^2; \tag{5}$$

$$F_{2,n}(x) = \{n^2x^{n+3}-(2n^2+2n-1)x^{n+2}+(n+1)^2x^{n+1}-x^2-x\}/(x-1)^3; \tag{6}$$

More generally (by induction or otherwise),

$$F_{m,n}(x) = \{x^{n+1}P_{m,n}(x)-P_{m,-1}(x)\}/(x-1)^{m+1}, \ m>0, \tag{7}$$

where $P_{m,n}(x)$ is a polynomial in x of degree m. We may suppose

$$P_{m,n}(x) = \sum_{r=0}^{m} A_r(m,n)x^r. \tag{8}$$

Note that $P_{0,n}(x)=1$, $P_{1,n}(x)=nx-(n+1)$, $P_{2,n}(x)=n^2x^2-(2n^2+2n-1)x+(n+1)^2$, etc. Repeated application of (3), using (7) and (8), yields the following general formula for $A_r(m,n)$:

$$A_r(m,n)=\sum_{s=r}^{m}(-1)^{s-r}{}_{m+1}C_{s-r}(n-m+s)^m. \tag{9}$$

Substituting the last expression into (8) yields the following development in the "umbral" calculus, involving the finite difference operators E and Δ (with operand z^m at indicated values of z):

$$\begin{aligned}P_{m,n}(x)&=\sum_{r=0}^{m}x^{m-r}\sum_{s=0}^{r}(-1)^s{}_{m+1}C_s(n-r+s)^m=x^m\left\{\sum_{s=0}^{m}(-E)^s{}_{m+1}C_s\sum_{r=s}^{m}(1/Ex)^r\right\}(z^m)\Big|_{z=n}\\&=x^m\left\{\sum_{s=0}^{m+1}(-E)^s{}_{m+1}C_s((Ex)^{-m-1}-(Ex)^{-s})/((Ex)^{-1}-1)\right\}(z^m)\Big|_{z=n}\\&=\left\{E^{-m}/(1-Ex)\cdot(1-E)^{m+1}-E/(1-Ex)\cdot(x-1)^{m+1}\right\}(z^m)\Big|_{z=n}\end{aligned}$$

Note that $(E-1)^{m+1}(z^m)=\Delta^{m+1}(z^m)=0$. Thus, since $E=1+\Delta$,

$$\begin{aligned}P_{m,n}(x)&=-(x-1)^{m+1}\{(1-xE)^{-1}\}(z^m)\Big|_{z=n+1}\\&=(x-1)^m\{(1+\Delta x/(x-1))^{-1}\}(z^m)\Big|_{z=n+1}.\end{aligned}$$

In particular,

$$P_{m,n}(\alpha)=\alpha^{-m}\{(1+\alpha^2\Delta)^{-1}\}(z^m)\Big|_{z=n+1}=\alpha^{-m}\sum_{s=0}^{m}(-1)^s\alpha^{2s}\Delta^s(z)^m\Bigg|_{z=n+1},$$

and likewise,

$$P_{m,n}(\beta)=\beta^{-m}\sum_{s=0}^{m}(-1)^s\beta^{2s}\Delta^s(z^m)\Bigg|_{z=n+1}.$$

Now, substituting these last results into the formula in (7), we obtain

$$\begin{aligned}F_{m,n}(\alpha)&=\alpha^{m+1}\left\{\alpha^{n+1-m}\sum_{s=0}^{m}(-1)^s\alpha^{2s}\Delta^s(n+1)^m-\alpha^{-m}\sum_{s=0}^{m}(-1)^s\alpha^{2s}\Delta^s(0)^m\right\}\\&=\sum_{s=0}^{m}(-1)^s\left\{\alpha^{n+2s+2}\Delta^s(n+1)^m-\alpha^{2s+1}\Delta^s(0)^m\right\},\end{aligned}$$

where, for brevity, we write $\Delta^s(a)^m$ for the more precise expression $\Delta^s(z)^m\big|_{z=a}$. Similarly, we obtain the following expression:

$$F_{m,n}(\beta)=\sum_{s=0}^{m}(-1)^s\{\beta^{n+2s+2}\Delta^s(n+1)^m-\beta^{2s+1}\Delta^s(0)^m\}.$$

We may now substitute these last expressions into (2) and obtain

$$\Sigma_{m,n}=\sum_{s=0}^{m}(-1)^s\{F_{n+2s+2}\Delta^s(n+1)^m-F_{2s+1}\Delta^s(0)^m\}. \tag{10}$$

This is still not in the form that is envisioned by the proposers, but it only takes a bit more effort to put it into such form; we resort once more to the umbral calculus. Returning to our earlier notation, we proceed as follows:

$$\begin{aligned}\alpha^m P_{m,n}(\alpha) &= \sum_{s=0}^{m}(-1)^s \alpha^{2s}\Delta^s(n+1)^m \\ &= \{(1-(-\alpha^2\Delta)^{m+1})/(1+\alpha^2\Delta)\}(n+1)^m \\ &= \{1/(1+\alpha^2\Delta)\}(n+1)^m.\end{aligned}$$

Then,

$$\begin{aligned}&5^{-1/2}\{\alpha^{m+n+2}P_{m,n}(\alpha)-\beta^{m+n+2}P_{m,n}(\beta)\} \\ &= 5^{-1/2}\{\alpha^{n+2}/(1+\alpha^2\Delta)-\beta^{n+2}/(1+\beta^2\Delta)\}(n+1)^m \\ &= \{(F_{n+2}+F_n\Delta)/(1+3\Delta+\Delta^2)\}(n+1)^m \\ &= \{(F_{n+1}+(1+\Delta)F_n)/(1+3\Delta+\Delta^2)\}(n+1)^m;\end{aligned}$$

finally, we may recast (10) as follows:

$$\Sigma_{m,n} = F_{n+1}G(\Delta)(n+1)^m + F_n G(\Delta)(n+2)^m - G(\Delta)(1)^m, \tag{11}$$

where

$$G(\Delta) = 1/(1+3\Delta+\Delta^2). \tag{12}$$

Comparing this with the desired format, we have

$$p_1^{(m)}(n) = G(\Delta)(n+1)^m,\ p_2^{(m)}(n) = p_1^{(m)}(n+1),\ C_m = -p_1^{(m)}(0). \tag{13}$$

Therefore, we may express $p_1^{(m)}$ and $p_2^{(m)}$, as well as C_m, in terms of essentially only one polynomial; thus,

$$\Sigma_{m,n} = F_{n+1}p^{(m)}(n) + F_n p^{(m)}(n+1) - p^{(m)}(0), \tag{14}$$

where $p^{(m)}(n) \equiv p_1^{(m)}(n)$, as given in (12) and (13).

This is a somewhat stronger statement than the initial conjecture, since it gives a more specific expression, relating the three quantities $p_1^{(m)}(n)$, $p_2^{(m)}(n)$, and C_m. Now we need to determine the coefficients of these polynomials. Let

$$p^{(m)}(n) = \sum_{i=0}^{m}(-1)^{m-i}a_i^{(m)}n^i;$$

note that $p^{(m)}(0) = (-1)^m a_0^{(m)} = -C_m$, which gives, essentially, Part 4 of the problem. Also, from (13), $p^{(m)}(n) = G(\Delta)(n+1)^m$. Now

$$\begin{aligned}G(\Delta) &= 1/(1+3\Delta+\Delta^2) = 5^{-1/2}\{\alpha^2/(1+\alpha^2\Delta)-\beta^2/(1+\beta^2\Delta)\} \\ &= \sum_{s=0}^{m}(-1)^s F_{2s+2}\Delta^s = \sum_{s=0}^{m}(-1)^s F_{2s+2}(E-1)^s = \sum_{s=0}^{m}(-1)^s F_{2s+2}\sum_{r=0}^{s}(-1)^r\,{}_sC_r\,E^{s-r};\end{aligned}$$

therefore,

$$p^{(m)}(n) = \sum_{s=0}^{m}(-1)^s F_{2s+2}\sum_{r=0}^{s}(-1)^r\,{}_sC_r(n+1+s-r)^m$$

$$=\sum_{s=0}^{m}(-1)^s F_{2s+2}\sum_{r=0}^{s}(-1)^r\ {}_sC_r\sum_{i=0}^{m} mC_i n^i(s+1-r)^{m-i}$$

$$=\sum_{i=0}^{m} n^i\ {}_mC_i\sum_{s=0}^{m}(-1)^s F_{2s+2}\sum_{r=0}^{s}(-1)^r\ {}_sC_r\, E^{s-r}(1)^{m-i}$$

$$=\sum_{i=0}^{m} n^i\ {}_mC_i\sum_{s=0}^{m}(-1)^s F_{2s+2}\Delta^s(1)^{m-i}=\sum_{i=0}^{m} n^i\ {}_mC_i\, G(\Delta)(1)^{m-i}.$$

Comparison of coefficients in the two expressions for $p^{(m)}(n)$ yields

$$a_i^{(m)}=(-1)^{m-i}\ {}_mC_i\, G(\Delta)(1)^{m-i}. \tag{15}$$

In similar fashion, we may obtain the following expression:

$$b_i^{(m)}=(-1)^{m-i}\ {}_mC_i\, G(\Delta)(2)^{m-i}. \tag{16}$$

We then see from (15) and (16) that $a_i^{(m+i)}/b_i^{(m+i)}=G(\Delta)(1)^m/G(\Delta)(2)^m$. However, we see that such result is independent of i, so we express such ratio as $\rho(m)$. Therefore, we have

$$a_i^{(m+i)}/b_i^{(m+i)}=a_0^{(m)}/b_0^{(m)}=\rho(m). \tag{17}$$

We now return to the problem of determining $\rho=\lim_{m\to\infty}\rho(m)$. First, however, we deduce some additional properties of the coefficients $a_i^{(m)}$ and $b_i^{(m)}$. From the expressions in (15) and (16), it readily follows that

$$a_i^{(m)}=m/i\,a_{i-1}^{(m-1)},\ b_i^{(m)}=m/i\,b_{i-1}^{(m-1)},\ i=1,2,\dots,m. \tag{18}$$

Then

$$p^{(m)}(n)=\sum_{i=0}^{m}(-1)^{m-i}a_i^{(m)}n^i=(-1)^m a_0^{(m)}+\sum_{i=1}^{m}(-1)^{m-i}m/i\,a_{i-1}^{(m-1)}n^i$$

$$=(-1)^m a_0^{(m)}+m\sum_{i=0}^{m-1}(-1)^{m-1-i}a_i^{(m-1)}(n^{i+1}/i+1).$$

We see then that

$$p^{(m)}(n)=(-1)^m a_o^{(m)}+m\int_0^n p^{(m-1)}(t)\,dt, \tag{19}$$

which is essentially Part 1 of the problem (stated somewhat more precisely).

We now return to the expression in (13), namely,

$$p^{(m)}(n)=G(\Delta)(n+1)^m. \tag{20}$$

Note that we may allow $n=0$ and $n=-1$ in this last expression, but that if $n\le -2$, extraneous and unintended terms arise that make the expression incorrect.

Note that $p^{(m)}(n+1)=(1+\Delta)p^{(m)}(n)$, while $p^{(m)}(n+2)=(1+\Delta)^2p^{(m)}(n)$; then $p^{(m)}(n+2)+p^{(m)}(n+1)=\{1+\Delta+1+2\Delta+\Delta^2\}p^{(m)}(n)=\{1+1+3\Delta+\Delta^2\}G(\Delta)(n+1)^m=\{1+G(\Delta)\}(n+1)^m$, or

$$p^{(m)}(n+2)+p^{(m)}(n+1)=(n+1)^m+p^{(m)}(n). \tag{21}$$

In particular, setting $n=-1$ (and assuming $m>0$),

$$p^{(m)}(1)+p^{(m)}(0)=p^{(m)}(-1). \tag{22}$$

Now observe that

$$p^{(m)}(-1) = (-1)^m \sum_{i=0}^{m} a_i^{(m)},$$

$$p^{(m)}(0) = (-1)^m a_0^{(m)} = p_2^{(m)}(-1) = (-1)^m \sum_{i=0}^{m} b_i^{(m)},$$

and

$$p^{(m)}(1) = p_2^{(m)}(0) = (-1)^m b_0^{(m)}.$$

From (22), we then deduce the following:

$$a_0^{(m)} + b_0^{(m)} = \sum_{i=0}^{m} a_i^{(m)} \quad \text{or} \quad b_0^{(m)} = \sum_{i=1}^{m} a_i^{(m)}, \text{ valid for } m > 0;$$

this is essentially Part 3 of the problem. Also

$$a_0^{(m)} = b_0^{(m)} + \sum_{i=1}^{m} b_i^{(m)} = \sum_{i=1}^{m} (a_i^{(m)} + b_i^{(m)}), \text{ valid for } m > 0;$$

this is essentially Part 2 of the problem.

That the $a_i^{(m)}$ and $b_i^{(m)}$ are integers is obvious from the formulas in (15) and (16). That they are positive requires more effort. An easy induction on (18) yields the following relations:

$$a_i^{(m)} = {}_mC_i a_0^{(m-i)}, \ b_i^{(m)} = {}_mC_i b_0^{(m-i)}, \ i = 0, 1, \ldots, m. \tag{23}$$

Thus, $a_i^{(m)}(b_i^{(m)}) > 0$ if and only if $a_0^{(m)}(b_0^{(m)}) > 0$, $m = 0, 1, \ldots$.

From (23) and Parts 2 and 3 of the problem, we see that, if $a_0^{(m)} > 0$, $m = 0, 1, 2, \ldots$, then $a_i^{(m)} > 0 \Rightarrow b_0^{(m)} > 0 \Rightarrow b_i^{(m)} > 0$ $(i = 0, 1, \ldots, m)$. Thus, it suffices to prove that $a_0^{(m)} > 0$, $m = 0, 1, \ldots$.

Our proof of this assertion is by induction (on m). The inductive step depends on the following recursive formula:

$$a_0^{(m+1)} = \sum_{i=0}^{[m/2]} a_{2i}^{(m)} / (2i+1), \ m = 0, 1, 2, \ldots. \tag{24}$$

If (24) is valid, our inductive hypothesis is that $a_0^{(m)} > 0$ for some $m \ge 0$. From our foregoing discussion, this implies that $a_{2i}^{(m)} > 0$, $i = 0, 1, \ldots, [m/2]$. Then (24) implies that $a_0^{(m+1)} > 0$. Since $a_0^{(0)} = 1$, this proves the hypothesis. Our task is thus reduced to proving (24).

We return to the results in (19) and (22). Expressing the integral recurrence in terms of the coefficients (and replacing m by $m+1$), we obtain

$$(-1)^m a_0^{(m+1)} = (m+1) \int_0^1 \sum_{i=0}^{m} (-1)^{m-i} \{1 + (-1)^i\} a_i^{(m)} t^i \, dt;$$

then

$$a_0^{(m+1)} = (m+1) \sum_{i=0}^{m} \{1 + (-1)^i\} a_i^{(m)} / (i+1),$$

which is equivalent to (24). □

Note: As noted, we allowed the values $n = 0$ and $n = -1$ although, in the original statement of the problem, n was required to be positive. It may be observed that if we substitute the values $n = 0$ or $n = -1$ in the formula given by (14), we find that the expression for $\Sigma_{m,n}$ dutifully vanishes, as we should expect from its original definition.

The only remaining task is to establish that $\rho = \alpha$, as conjectured by the proposers.

We write a_m for $a_0^{(m)}$ and b_m for $b_0^{(m)}$. From (18), we easily deduce the following recursive formulas:

$$a_m = \sum_{i=0}^{m-1} {}_mC_i(a_i + b_i),\ b_m = \sum_{i=0}^{m-1} {}_mC_i\, a_i,\ m = 1, 2, \ldots. \tag{25}$$

We may also express a_m and b_m in the umbral calculus as follows:

$$a_m = (-1)^m G(\Delta)(1)^m,\ b_m = (-1)^{m+1}\{\Delta G(\Delta)\}(0)^m. \tag{26}$$

It is apparent that a_m and b_m are unbounded (as $m \to \infty$). Also note that $0 < b_m < a_m$ for all $m \ge 1$. Then (25) implies that $1 < \rho(m) \le 2$ for all $m \ge 1$. From the expressions for a_m and b_m in (26), it is not difficult to show (by expansion of the operators) that ρ exists.

We may express (25) in the following form:

$$a_m = \sum_{i=0}^{m-1} {}_mC_i(\rho(i)+1)b_i,\ b_m = \sum_{i=0}^{m-1} {}_mC_i\, \rho(i) b_i,\ m = 1, 2, \ldots. \tag{27}$$

Hence, $\rho(m) = 1 + 1/\rho(m-1)$, where $\rho(m-1)$ is a "weighted average" ρ, and the "weights" are the quantities ${}_mC_i\, b_i$ in the sums. Letting $m \to \infty$, we may therefore deduce that $1 \le \rho \le 2$ and $\rho = 1 + \rho^{-1}$ This, in turn, implies that $\rho = \alpha$.

For additional confirmation, the simple continued fraction (s.c.f.) expansion for $\rho(m)$ approaches the infinite s.c.f. $[1, 1, 1, \ldots]$, which is known to be the s.c.f. for α. □

Also solved by I. Strazdins.

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