EULERIAN NUMBERS: INVERSION FORMULAS AND CONGRUENCES MODULO A PRIME

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1. INTRODUCTION

The identity

$$n^m = \sum_{k=1}^{m} C_k^m \binom{n+k-1}{m}, \tag{1}$$

where the C_k^m's are *Eulerian Numbers*, was known by Worpitsky [16]. Eulerian numbers satisfy the recursive relation

$$C_k^m = (m+1-k)\,C_{k-1}^{m-1} + kC_k^{m-1} \tag{2}$$

for $2 \le k \le m-1$ and $C_1^m = C_m^m = 1$. This can be proved by induction, see Stanley [13], and is also proved by Carlitz [4], [5] and by Krishnapriyan [11] in different ways.

n^m can also be expressed as a linear combination of binomial coefficients in the form

$$n^m = \sum_{k=1}^{m} D_k^m \binom{n}{k}, \tag{3}$$

where the D_k^m coefficients satisfy the recursive relation

$$D_k^m = k(D_{k-1}^{m-1} + D_k^{m-1}) \tag{4}$$

for $2 \le k \le m-1$; also $D_1^m = 1$ and $D_m^m = m!$, which is not difficult to check using induction. The uniqueness of the C and D in formulas (1) and (3) is also easily proved.

The C numbers form the following triangular array:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1 \\
 & & & & \cdots & & & &
\end{array}$$

where the *Pascal-like* formation rule is given by formula (2). The D numbers form the array:

$$\begin{array}{lllll}
1 & & & & \\
1 & 2 & & & \\
1 & 6 & 6 & & \\
1 & 14 & 36 & 24 & \\
1 & 30 & 150 & 240 & 120 \\
 & & \cdots & &
\end{array}$$

where the formation rule for this table is given by formula (4). The D numbers are related to the *Stirling Numbers of the Second Kind* S_k^m by the expression $D_k^m = k!\,S_k^m$, which is easy to prove by considering the recursive relation

$$S_k^m = kS_k^{m-1} + S_{k-1}^{m-1}.$$

Thus, we can deduce

$$n^m = \sum_{k=1}^{m} k!\,S_k^m \binom{n}{k},$$

which is studied in several combinatorics textbooks, (see, e.g., Aigner [1] or Stanley [13]).

We can use expressions (1) and (3) to obtain formulas for the sum of powers $S_m(n) = 1^m + 2^m + \cdots + n^m$. Adding terms, and using the identity

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1},$$

we deduce

$$S_m(n) = \sum_{k=1}^{m} C_k^m \binom{n+k}{m+1} \tag{5}$$

and

$$S_m(n) = \sum_{k=1}^{m} D_k^m \binom{n+1}{k+1}, \tag{6}$$

where the C and D coefficients are defined by (2) and (4). Many papers have been written concerning formulas for $S_m(n)$. Perhaps the best known formulas express this sum as a polynomial in n of degree $m+1$ with coefficients involving Bernoulli numbers. See, for example, the papers by Christiano [6] and by de Bruyn and de Villiers [7]. Burrows and Talbot [2] treat this sum as a polynomial in $(n+1/2)$, and Edwards [8] expresses the sums $S_m(n)$ as polynomials in $\sum k$ and $\sum k^2$. Formulas (5) and (6) express this sum as linear combinations of binomial coefficients. Formula (5) is also discussed by Graham et al. [9]; Shanks [12] deduces (5) by considering sums of powers of binomial coefficients. Hsu [10] obtains formula (6) by studying sums of the form $\sum_{k=0}^{n} F(n,k)k^p$ for different functions $F(n,k)$ and expresses these sums as linear combinations of the D_k^m coefficients.

The combinatorial significance of Eulerian numbers is known. In Section 2 we discuss a combinatorial meaning of D_k^m and deduce some nonrecursive formulas for both C and D numbers by combinatorial means.

In Section 3 we show that the C and D numbers satisfy the inversion formulas

$$D_k^m = \sum_{i=0}^{k-1} \binom{m-k+i}{i} C_{k-i}^m$$

and

$$C_k^m = \sum_{i=0}^{k-1} (-1)^i \binom{m-k+i}{i} D_{k-i}^m.$$

We then use these to obtain a number-theoretical result analogous to the well-known fact that

$$p \,\Big|\, \binom{p}{k}$$

whenever p is a prime and $1 \le k \le p-1$.

2. COMBINATORIAL MEANING

The combinatorial significance of the Eulerian numbers is known. C_k^m is the number of permutations $p_1p_2\dots p_m$ of $\{1, 2, \dots, m\}$ that have $k-1$ *ascents* [9, pp. 253-58], that is, $k-1$ places where $p_j < p_{j+1}$.

A combinatorial meaning of the D numbers is given by the following proposition.

Proposition 2.1: D_k^m is the number of surjective functions from the set $\{1, 2, \dots, m\}$ onto the set $\{1, 2, \dots, k\}$.

Proof: Consider the number of m-tuples $(a_1, a_2, \dots, a_m)$, where $1 \le a_i \le n$, $i = 1, 2, \dots, m$. We have a total of n^m different m-tuples.

Now, the total number of different m-tuples is equal to the number of m-tuples whose elements are equal plus the number of m-tuples whose elements are two different numbers, and so on.

Since the number of subsets of k elements of a set of n elements is given by $\binom{n}{k}$, the number of m-tuples whose elements are k different numbers is $E_k^m\binom{n}{k}$, where E_k^m is the number of m-tuples with k different numbers, which is equal to the number of surjective functions from $\{1, 2, \dots, m\}$ onto $\{1, 2, \dots, k\}$. Hence,

$$n^m = \sum_{k=1}^{m} E_k^m \binom{n}{k}.$$

By unicity of the D_k^m, we conclude that $D_k^m = E_k^m$.

We shall now deduce a formula for D_k^m.

Proposition 2.2: The number D_k^m is given by

$$D_k^m = \sum \frac{m!}{x_1!x_2!\dots x_k!}, \tag{7}$$

where the sum is taken over all the positive integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = m. \tag{8}$$

Proof: By Proposition 2.1, D_k^m is the number of surjective functions from $\{1, 2, \dots, m\}$ onto $\{1, 2, \dots, k\}$, so we count the m-tuples formed "using" all the numbers $1, 2, \dots, k$.

To form an m-tuple with the numbers $1, 2, \dots, k$, we use the number 1 x_1 times, the number 2 x_2 times, and so on up to the number k x_k times, so that $x_1 + x_2 + \cdots + x_k = m$ and $x_i \ge 1$ for $i = 1, 2, \dots, k$. For each solution to this equation, we have

$$\frac{m!}{x_1!x_2!\dots x_k!}$$

ways of ordering the numbers $1, 2, \dots, k$ in the m-tuple. Therefore,

$$D_k^m = \sum \frac{m!}{x_1!x_2!\dots x_k!},$$

where the sum is taken over all positive integer solutions of equation (8).

Note that the expression

$$\frac{m!}{x_1!x_2!\dots x_k!}$$

is equal to the multinomial coefficient

$$\binom{m}{x_1, x_2, \dots, x_k},$$

which is the coefficient of $a_1^{x_1}a_2^{x_2}\dots a_k^{x_k}$ in the expansion of $(a_1+a_2+\dots+a_k)^m$. For a discussion of multinomial expansions, see Tomescu [14, p. 17]. Thus, we have the expression

$$D_k^m = \sum_{x_i \ge 1} \binom{m}{x_1, x_2, \dots, x_k}.$$

Other formulas for both D_k^m and C_k^m can be obtained. The expression

$$D_k^m = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^m$$

can also be obtained by counting functions from $\{1, 2, \dots, m\}$ onto $\{1, 2, \dots, k\}$, see [14, pp. 41-48]. The expression

$$C_k^m = \sum_{j=0}^{k-1} (-1)^j \binom{m+1}{j} (k-j)^m$$

for the Eulerian numbers appears in papers by Carlitz [4], [5], and by Velleman and Call [15].

We are particularly interested in formula (7) because, from it, we can easily deduce that if p is a prime, then

$$D_k^p \equiv \begin{cases} 1 \pmod{p} & \text{if } k=1, \\ 0 \pmod{p} & \text{if } 2 \le k \le p. \end{cases} \tag{9}$$

We will use this result in Section 4.

3. INVERSION FORMULAS

In this section we discuss inversion formulas between the C and D numbers discussed above. For the purposes of this section, let us extend the definition of the C and D numbers by

$$C_k^m = \begin{cases} (m+1-k)C_{k-1}^{m-1} + kC_k^{m-1} & \text{if } 1 \le k \le m, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

and

$$D_k^m = \begin{cases} k(D_{k-1}^{m-1} + D_k^{m-1}) & \text{if } 1 \le k \le m, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

with $C_1^1 = D_1^1 = 1$. Clearly, these formulas are extensions of (2) and (4) above. For the proof of the next theorem, we will use the following identities:

$$(m+1)\binom{m+1-k+i}{i} - (i+1)\binom{m+1-k+i}{i+1} = k\binom{m+1-k+i}{i}; \tag{12}$$

$$(i+1)\binom{m+1-k+i}{i+1}+\binom{m+1-k+i}{i}=(m-k+2)\binom{m+1-k+i}{i}; \tag{13}$$

$$\sum_{i=0}^{k}(-1)^i\binom{n-i}{k-i}\binom{n}{i}=\begin{cases}1 & \text{if } k=0,\\ 0 & \text{if } 1\le k\le n.\end{cases} \tag{14}$$

Expressions (12) and (13) are easy to verify using the definition of the binomial coefficient, while expression (14) is proven by induction (see [14, p. 20, prob. 2.15]).

Now we state the following theorem.

Theorem 3.1: The numbers C_k^m and D_k^m are related by the inversion formulas

$$D_k^m=\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_{k-i}^m \tag{15}$$

and

$$C_k^m=\sum_{i=0}^{k-1}(-1)^i\binom{m-k+i}{i}D_{k-i}^m. \tag{16}$$

Proof: We first prove (15). This will be done by induction on m. It is easy to verify that formula (15) is true for $m=1$, so assume it is true for some m. We need to show that

$$\sum_{i=0}^{k-1}\binom{m+1-k+i}{i}C_{k-i}^{m+1}=D_k^{m+1}.$$

In order to simplify the notation, let

$$\mathscr{C}=\sum_{i=0}^{k-1}\binom{m+1-k+i}{i}C_{k-i}^{m+1},\ C_0=C_{k-i}^m,\ \text{and}\ C_1=C_{k-i-1}^m.$$

By (10), properties of sums, and Pascal's identity,

$$\begin{aligned}
\mathscr{C}&=\sum_{i=0}^{k-1}\binom{m+1-k+i}{i}[(k-i)C_0+(m+2-k+i)C_1]\\
&=\sum_{i=0}^{k-1}(k-i)\binom{m+1-k+i}{i}C_0+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1\\
&=\sum_{i=0}^{k-1}(k-i)\left[\binom{m-k+i}{i}+\binom{m-k+i}{i-1}\right]C_0+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1\\
&=k\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_0-\sum_{i=1}^{k-1}i\binom{m-k+i}{i}C_0\\
&\quad+\sum_{i=1}^{k-1}(k-i)\binom{m-k+i}{i-1}C_0+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1\\
&=k\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_0-\sum_{i=0}^{k-2}(i+1)\binom{m-k+i+1}{i}C_1\\
&\quad+\sum_{i=0}^{k-2}(k-i-1)\binom{m-k+i+1}{i}C_1+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1
\end{aligned}$$

$$= k\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_0 + \sum_{i=0}^{k-2}\left[(m+1)\binom{m+1-k+i}{i} - (i+1)\binom{m+1-k+i}{i+1}\right]C_1.$$

Using identity (12),

$$\mathscr{C} = k\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_0 + k\sum_{i=0}^{k-2}\binom{m+1-k+i}{i}C_1.$$

Finally, by our induction hypothesis and (11),

$$\mathscr{C} = k(D_k^m + D_{k-1}^m) = D_k^{m+1}.$$

Similar inductive reasoning, using identity (13), proves formula (16). Another way to prove (16) is by expressing relations (15) and (16) in matrix form, $\mathbf{Cc} = \mathbf{d}$ and $\mathbf{Dd} = \mathbf{c}$, where

$$\mathbf{c} = \begin{pmatrix} C_1^m \\ C_2^m \\ \vdots \\ C_m^m \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} D_1^m \\ D_2^m \\ \vdots \\ D_m^m \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} \binom{m-1}{0} & 0 & \cdots & 0 \\ \binom{m-1}{1} & \binom{m-2}{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m-1}{m-1} & \binom{m-2}{m-2} & \cdots & \binom{0}{0} \end{pmatrix},$$

and

$$\mathbf{D} = \begin{pmatrix} \binom{m-1}{0} & 0 & \cdots & 0 \\ -\binom{m-1}{1} & \binom{m-2}{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1}\binom{m-1}{m-1} & (-1)^{m-2}\binom{m-2}{m-2} & \cdots & \binom{0}{0} \end{pmatrix},$$

and verifying that $\mathbf{CD} = \mathbf{I}_m$, where $\mathbf{I}_m$ is the $m \times m$ identity matrix.

The i, j^{th} term of $\mathbf{C}$ is given by

$$c_{ij} = \binom{m-j}{i-j},$$

while the i, j^{th} term of $\mathbf{D}$ is given by

$$d_{ij} = (-1)^{i-j}\binom{m-j}{i-j}.$$

Hence, the i, j^{th} term of $\mathbf{CD}$ is given by

$$\sum_{k=1}^{m} c_{ik}d_{kj} = \sum_{k=1}^{m}(-1)^{k-j}\binom{m-k}{i-k}\binom{m-j}{k-j}.$$

If we substitute r for $k-j$, the right-hand side becomes

$$\sum_{r=1-j}^{m}(-1)^r\binom{(m-j)-r}{(i-j)-r}\binom{m-j}{r}. \tag{17}$$

Now, if $r<0$, then $\binom{m-j}{r}=0$, and if $r>i-j$, then $\binom{(m-j)-r}{(i-j)-r}=0$. Hence, expression (17) becomes

$$\sum_{r=0}^{i-j}(-1)^r\binom{(m-j)-r}{(i-j)-r}\binom{m-j}{r},$$

which, by (14), is equal to 0 for $j+1\le i\le m$, and equal to 1 if $i=j$. It is understood that this sum is 0 in the case $j>i$. Therefore, $\mathbf{CD}=\mathbf{I}_m$.

4. CONGRUENCES MODULO A PRIME

Now let us go back to our result (9), which stated that, if p is a prime, then

$$D_k^p\equiv\begin{cases}1 \pmod p & \text{if } k=1,\\ 0 \pmod p & \text{if } 2\le k\le p.\end{cases} \tag{18}$$

For instance, the fifth row of the table formed by the D numbers is 1, 30, 150, 240, 120, and we see that all these numbers, except for the first one, which is equal to 1, are multiples of 5. This is analogous to the well-known fact that, if p is a prime number, then

$$\binom{p}{k}\equiv\begin{cases}1 \pmod p & \text{if } k=0 \text{ or } k=p,\\ 0 \pmod p & \text{if } 1\le k\le p-1.\end{cases} \tag{19}$$

We will prove that statement (18) is equivalent to the statement

$$C_k^p\equiv 1 \pmod p \tag{20}$$

whenever p is a prime and $1\le k\le p$. For instance, in the fifth row of the table formed by the Eulerian numbers, 1, 26, 66, 26, 1, all the numbers are congruent to 1 modulo 5.

For the proof of the equivalence of these two statements, we will use the following identity,

$$\sum_{i=0}^{k-1}\binom{n-k+i}{i}=\binom{n}{k-1}, \tag{21}$$

which is not difficult to verify by induction on n, together with the statement,

$$(-1)^{k-1}\binom{p-1}{k-1}\equiv 1 \pmod p, \tag{22}$$

which is easy to show using (19) and *Pascal's identity* (see [3, p. 96. prob. 12]).

Now we prove the equivalence of statements (18) and (20), which we state as a theorem.

Theorem 4.1: If p is a prime, then statements (18) and (20) are equivalent.

Proof: Assume that (20) is true. For $k=1$, $D_k^p=1\equiv 1 \pmod p$. By Theorem 3.1, we have

$$D_k^p=\sum_{i=0}^{k-1}\binom{p-k+i}{i}C_{k-i}^p.$$

Then, using (20) and identity (21), for $2\le k\le p$,

$$D_k^p \equiv \sum_{i=0}^{k-1} \binom{p-k+i}{i} = \binom{p}{k-1} \equiv 0 \pmod{p}.$$

Conversely, assume statement (18) is true. By Theorem 3.1 and (22),

$$C_k^p = \sum_{i=0}^{k-1} (-1)^i \binom{p-k+i}{i} D_{k-i}^p \equiv (-1)^{k-1} \binom{p-1}{k-1} \equiv 1 \pmod{p}.$$

Theorem 4.1 and the validity of (18) imply the validity of (20). We see that this theorem, together with Theorem 3.1, shows a strong relationship between the two sets of numbers C_k^m and D_k^m. We expect this relationship to have a combinatorial significance as well.

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