A GENERALIZATION OF JACOBSTHAL POLYNOMIALS

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1. INTRODUCTION

In a recent article, André-Jeannin [1] introduced a generalization of the Morgan-Voyce polynomials by defining the sequence of polynomials $\{P_n^{(r)}(x)\}$ by the relation

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x) \quad (n \geq 2), \tag{1a}$$

with

$$P_0^{(r)}(x) = 1 \quad \text{and} \quad P_1^{(r)}(x) - x + r + 1. \tag{1b}$$

Subsequently, Horadam [2] defined a closely related sequence of polynomials $\{Q_n^{(r)}(x)\}$ by the relation

$$Q_n^{(r)}(x) = (x+2)Q_{n-1}^{(r)}(x) = Q_{n-2}^{(r)}(x) \quad (n \geq 2), \tag{2a}$$

with

$$Q_0^{(r)}(x) = 1 \quad \text{and} \quad Q_1^{(r)}(x) = x + r + 2. \tag{2b}$$

They also established that

$$P_n^{(0)}(x) = b_n(x), \tag{3a}$$

$$P_n^{(1)}(x) = B_n(x), \tag{3b}$$

and

$$Q_n^{(0)}(x) = C_n(x), \tag{3c}$$

where $b_n(x)$ and $B_n(x)$ are the classical Morgan-Voyce polynomials defined in [6] and $C_n(x) = 2c_n(x)$, where $c_n(x)$ is the polynomial introduced by Swamy and Bhattacharyya [7] in the analysis of ladder networks. It has also been established in [1] and [2] that, if

$$P_n^{(r)}(x) = \sum_{k \geq 0} a_{n,k}^{(r)} x^k \tag{4a}$$

and

$$Q_n^{(r)}(x) = \sum_{k \geq 0} b_{n,k}^{(r)} x^k, \tag{4b}$$

then, for any $n \geq 0$, $k \geq 0$,

$$a_{n,k}^{(r)} = \binom{n+k}{2k} + r\binom{n+k}{2k+1} \tag{5a}$$

and

$$b_{n,k}^{(r)} = \frac{n}{k}\binom{n-1+k}{2k-1} + r\binom{n+k}{2k+1}. \tag{5b}$$

The purpose of this short article is to introduce two new sequences of polynomials $\{p_n^{(r)}(x)\}$ and $\{q_n^{(r)}(x)\}$, and then relate them to Jacobsthal polynomials (see [3] and [5]).

2. THE NEW POLYNOMIALS $\{p_n^{(r)}(x)\}$ AND $\{q_n^{(r)}(x)\}$

Let us define the following two sequences of polynomials:

$$p_n^{(r)}(x) = p_{n-1}^{(r)}(x) + xp_{n-2}^{(r)}(x) \quad (n \geq 2) \tag{6a}$$

with

$$p_0^{(r)}(x) = 1 \text{ and } p_1^{(r)}(x) = r, \tag{6b}$$

and

$$q_n^{(r)}(x) = q_{n-1}^{(r)}(x) + xq_{n-2}^{(r)}(x) \quad (n \geq 2) \tag{7b}$$

with

$$q_0^{(r)}(x) = 2 \text{ and } q_1^{(r)}(x) = r+1, \tag{7b}$$

where r is a real number.

If we now express $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$ by

$$p_n^{(r)}(x) = \sum_{k\geq 0} c_{n,k}^{(r)} x^k \tag{8a}$$

and

$$q_n^{(r)}(x) = \sum_{k\geq 0} d_{n,k}^{(r)} x^k, \tag{8b}$$

we can obtain recurrence relations for $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$, and derive expressions for $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$ using the procedures adopted in [1] and [2]. However, we will use the properties of the sequence $w_n(a, b; p, q)$ defined by Horadam [4] to obtain a direct expression for $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$. From [4], we know that the solution $w_n(a, b; x)$ of the equation

$$w_n(x) = w_{n-1}(x) + xw_{n-2}(x) \quad (n \geq 2) \tag{9a}$$

with

$$w_0(x) = a \text{ and } w_1(x) = b \tag{9b}$$

is given by

$$w_n(x) = w_1(x)u_{n-1}(x) + xw_0(x)u_{n-2}(x), \tag{10a}$$

where

$$u_n(x) = w_n(1, 1; x). \tag{10b}$$

Hence, from (6), (7), (9), and (10), we have

$$p_n^{(r)}(x) = ru_{n-1}(x) + xu_{n-2}(x) = u_n(x) + (r-1)u_{n-1}(x) \tag{11a}$$

and

$$q_n^{(r)}(x) = (r+1)u_{n-1}(x) + 2xu_{n-2}(x) = u_n(x) + xu_{n-2}(x) + ru_{n-1}(x). \tag{11b}$$

3. $p_n^{(r)}(x)$, $q_n^{(r)}(x)$ AND JACOBSTHAL POLYNOMIALS

We now observe that

$$u_n(x) = J_{n+1}(x) \tag{12a}$$

and

$$v_n(x) = j_n(x), \tag{12b}$$

where $J_n(x)$ and $j_n(x)$ are the Jacobsthal polynomials (see [3]). It is to be noted that we have used x instead of $(2x)$ in the definitions of $J_n(x)$ and $j_n(x)$; that is, $J_n(x)$ and $j_n(x)$ are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \quad J_o(x) = 0, \ J_o(x) = 1, \tag{13a}$$

and

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x), \quad j_o(x) = 2, \ j_o(x) = 1. \tag{13b}$$

Hence, from (11), (12), and (13), we get

$$p_n^{(r)}(x) = J_{n+1}(x) + (r-1)J_n(x) \tag{14a}$$

and

$$q_n^{(r)}(x) = j_n(x) + rJ_n(x), \tag{14b}$$

where we have used the relation

$$j_n(x) = J_{n+1}(x) + xJ_{n-1}(x). \tag{15}$$

Using the closed-form expressions for $J_n(x)$ and $j_n(x)$ derived in [3], we can derive polynomial expressions for $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$ and, thus, expressions for $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$. Also, the interrelationship between $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$ may be expressed in terms of the following relation, which is a consequence of (14a) and (14b):

$$q_n^{(r)}(x) = p_n^{(r)}(x) + J_n(x) + xJ_{n-1}(x). \tag{16}$$

Of course, we have the following particular cases:

$$p_n^{(0)}(x) = xJ_{n-1}(x), \tag{17a}$$

$$p_n^{(1)}(x) = J_{n+1}(x), \tag{17b}$$

$$q_n^{(0)}(x) = j_n(x), \tag{17c}$$

$$q_n^{(1)}(x) = 2J_{n+1}(x). \tag{17d}$$

We may derive other relations between $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$ by utilizing the properties of $J_n(x)$ and $j_n(x)$. However, we will not pursue them here.

4. THE POLYNOMIALS $P_n^{(r)}(x)$ AND $Q_n^{(r)}(x)$

As was done in Section 2, by using the results of Horadam [4] concerning the generalized Fibonacci sequence, we may show that

$$P_n^{(r)}(x) = U_{n+1}(x) + (r-1)U_n(x) \tag{18a}$$

and

$$Q_n^{(r)}(x) = V_n(x) + rU_n(x), \tag{18b}$$

where

$$U_n(x) = w_n(0, 1; x), \tag{19a}$$

$$V_n(x) = w_n(2, x+2; x), \tag{19b}$$

and $w_n(a, b; x)$ is now the solution of the equation

$$w_n(x) = (x+2)w_{n-1}(x) - w_{n-2}(x) \quad (n \geq 2) \tag{20a}$$

with

$$w_0(x) = a \quad \text{and} \quad w_1(x) = b. \tag{20b}$$

From the properties of the polynomials $B_n(x)$ and $C_n(x)$ given in [6] and [7], we can relate $B_n(x)$ and $C_n(x)$ directly to $U_n(x)$ and $V_n(x)$ by

$$B_n(x) = U_{n+1}(x) \tag{21a}$$

and

$$C_n(x) = V_n(x). \tag{21b}$$

Thus, we have the following relations for $n \geq 1$:

$$P_n^{(r)}(x) = B_n(x) + (r-1)B_{n-1}(x), \tag{22a}$$

$$Q_n^{(r)}(x) = C_n(x) + rB_{n-1}(x), \tag{22b}$$

and

$$Q_n^{(r)}(x) = P_n^{(r)}(x) + b_{n-1}(x), \tag{22c}$$

where we have used the relations (see [7])

$$b_n(x) = B_n(x) - B_{n-1}(x) \tag{23a}$$

and

$$C_n(x) = B_n(x) - B_{n-2}(x). \tag{23b}$$

It can be observed that relations (3a), (3b), and (3c) directly follow from (22a) and (22b).

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