ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-878 ***Proposed by L. A. G. Dresel, Reading, England***

Show that, for positive integers n, the harmonic mean of F_n and L_n can be expressed as the ratio of two Fibonacci numbers, and that it is equal to $L_{n-1} + R_n$ where $|R_n| \le 1$. Find a simple formula for R_n.

Note: If h is the harmonic mean of x and y, then $2/h = 1/x + 1/y$.

B-879 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***

Let $\langle c_n \rangle$ be defined by the recurrence $c_{n+4} = 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n$ with initial conditions $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, and $c_3 = 6$. Express c_n in terms of Fibonacci and/or Lucas numbers.

B-880 ***Proposed by A. J. Stam, Winsum, The Netherlands***

Express

$$\sum_{2i \le m} \binom{m-i}{i} (-1)^i 3^{m-2i}$$

in terms of Fibonacci and/or Lucas numbers.

B-881 ***Proposed by Mohammad K. Azarian, University of Evansville, IN***

Consider the two equations

$$\sum_{i=1}^{n} L_i x_i = F_{n+3} \quad \text{and} \quad \sum_{i=1}^{n} L_i y_i = L_2 - F_{n+1}.$$

Show that the number of positive integer solutions of the first equation is equal to the number of nonnegative integer solutions of the second equation.

B-882 ***Proposed by A. J. Stam, Winsum, The Netherlands***

Suppose the sequence $\langle A_n \rangle$ satisfies the recurrence $A_n = A_{n-1} + A_{n-2}$. Let

$$B_n = \sum_{k=0}^{n} (-1)^k A_{n-2k}.$$

Prove that $B_n = A_0 F_{n+1}$ for all nonnegative integers n.

B-883 ***Proposed by L. A. G. Dresel, Reading, England***

Let $\langle f_n \rangle$ be the Fibonacci sequence F_n modulo p, where p is a prime, so that we have $f_n \equiv F_n \pmod{p}$ and $0 \le f_n < p$ for all $n \ge 0$. The sequence $\langle f_n \rangle$ is known to be periodic. Prove that, for a given integer c in the range $0 \le c < p$, there can be at most four values of n for which $f_n = c$ within any one cycle of this period.

SOLUTIONS

A Perfect Square

B-860 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 36, no. 5, November 1998)

Let k be a positive integer. The sequence $\langle A_n \rangle$ is defined by the recurrence $A_{n+2} = 2kA_{n+1} - A_n$ for $n \ge 0$ with initial conditions $A_0 = 0$ and $A_1 = 1$. Prove that $(k^2-1)A_n^2 + 1$ is a perfect square for all $n \ge 0$.

Solution by Don Redmond, Southern Illinois University, Carbondale, IL

We give a generalization. Let p and q be integers and let $A_0 = 0$ and $A_1 = 1$. Define, for $n \ge 0$, the sequence $\langle A_n \rangle$ by $A_{n+2} = 2pA_{n+1} - qA_n$. Then, for $n \ge 0$,

$$q^n + (p^2 - q)\, A_n^2$$

is a perfect square. If we let $q = 1$ and $p = k$, we obtain the desired result.

Let s and t be the roots of the polynomial $x^2 - 2px + q = 0$. Then we know that we can write, for $n \ge 0$,

$$A_n = \frac{s^n - t^n}{s - t}.$$

Now $s + t = 2p$, $st = q$, and $s - t = 2\sqrt{p^2 - q}$. Thus,

$$4q^n + 4(p^2 - q)\, A_n^2 = 4(st)^n + (s-t)^2 \left(\frac{s^n - t^n}{s - t} \right)^2 = 4(st)^n + (s^n - t^n)^2 = (s^n + t^n)^2.$$

Hence

$$q^n + (p^2 - q)\, A_n^2 = \left(\frac{s^n + t^n}{2} \right)^2.$$

Finally, $(s^n + t^n)/2$ is indeed an integer because it satisfies the same recurrence as A_n but with initial values 1 and p.

Seiffert also found this generalization. Lord showed that, for the original sequence,

$$(k^2-1)A_n^2+1=(kA_n-A_{n-1})^2.$$

Redmond noted that another generalization can be found on page 501 of [1]: If a_0 is any integer, $a_1=ka_0+p$ and, for $n\ge 0$, $a_{n+2}=2ka_{n+1}-a_n$, then $(k^2-1)(a_n^2-a_0^2)+p^2$ is a perfect square. The problem at hand is the case $a_0=0$ and $p=1$.

Reference:

1. Don Redmond. *Number Theory: An Introduction*. New York: Marcel Dekker, 1996.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Steve Edwards, N. Gauthier, Joe Howard, Hans Kappus, Harris Kwong, Graham Lord, Maitland A. Rose, H.-J. Seiffert, Indulis Strazdins, Andràs Szilàrd, and the proposer.

Integer Coefficients?

B-861 ***Proposed by the editor***
(Vol. 36, no. 5, November 1998)

The sequence $w_0, w_1, w_2, w_3, w_4, \ldots$ satisfies the recurrence $w_n=Pw_{n-1}-Qw_{n-2}$ for $n>1$. If every term of the sequence is an integer, must P and Q both be integers?

Counterexample by Steve Edwards, Southern Polytechnic State University, Marietta, GA

The sequence $w_n=k$, where k is an integer, is a counterexample when p is not an integer and $P-Q=1$.

Solution by L. A. G. Dresel, Reading, England

We shall prove that P and Q must both be integers provided that $w_1^2-w_0w_2\ne 0$.

Let $D_n=w_{n+1}^2-w_nw_{n+2}$. Eliminating P from the equations $w_{n+2}=Pw_{n+1}-Qw_n$ and $w_{n+3}=Pw_{n+2}-Qw_{n+1}$, we have $D_{n+1}=QD_n$ for $n\ge 0$. Therefore, $D_1=QD_0$, and by induction $D_n=Q^nD_0$ for $n\ge 0$.

If $D_0\ne 0$, then $Q=D_1/D_0$ is the ratio of two integers, and then $D_n=Q^nD_0$ for all $n\ge 1$ implies that Q must be an integer.

It remains to prove that P must also be an integer in this case. Suppose, on the contrary, that P is a rational fraction, $P=p/d$, where $\gcd(p,d)=1$. Consider the recurrence in the form $Pw_n=w_{n+1}+Qw_{n-1}$ for $n\ge 1$. It follows that d divides w_n, so that d divides $w_1, w_2, w_3, \ldots$. Therefore, for $n\ge 2$, the right side of the recurrence is divisible by d, and we have d^2 divides w_2, w_3, $w_4, \ldots$. Continuing in this way (let us call it the escalator principle), we find that for each n, d^n divides w_n. Hence d^{2n+2} divides $D_n=Q^nD_0$ for all n, and it follows that Q is divisible by d^2.

Returning again to the recurrence $Pw_n=w_{n+1}+Qw_{n-1}$ for $n\ge 1$, we see that the right side is divisible by d^2, and therefore d^3 divides $w_1, w_2, w_3, \ldots$. Then, for $n\ge 2$, the right side of the recurrence is divisible by d^5, so that d^6 divides $w_2, w_3, w_4, \ldots$. Continuing with this escalator principle, we find that, for each n, d^{3n} divides w_n. Hence, d^{6n+6} divides $D_n=Q^nD_0$ for all n, and it follows that Q is divisible by d^6. Returning again (and again) to the recurrence formula and applying the escalator principle, we require even higher powers of d dividing both Q and D_0, so that we cannot construct the sequence of integers unless $d=1$.

This implies that when $D_0\ne 0$, both P and Q must be integers.

Counterexamples also received by Richard André-Jeannin, Paul S. Bruckman, H.-J. Seiffert, and Andràs Szilàrd.

Large LCM

B-862 ***Proposed by Charles K. Cook, University of South Carolina, Sumter, SC (Vol. 36, no. 5, November 1998)***

Find a Fibonacci number and a Lucas number whose sum is 114,628 and whose least common multiple is 567,451,586.

Solution by Scott H. Brown, Auburn University at Montgomery, Montgomery, AL

Since the sum of the two numbers ends in 8, we test the combinations in the one's digits: $0+8$, $1+7$, $2+6$, $3+5$, and $4+4$. Testing these combinations and observing that they must add up to 114,628, many of these combinations are eliminated with the exception of the following:

(a) $F_{21} = 10{,}946 \quad L_{24} = 103{,}682$;

(b) $F_{25} = 75{,}025 \quad L_{22} = 39{,}603$.

These values were found on pages 83 and 84 in [1].

Factoring the integers in question, we find $10946 = 2\cdot 13\cdot 421$, $103682 = 2\cdot 47\cdot 1103$, $75025 = 5^2\cdot 3001$, and $39603 = 3\cdot 43\cdot 307$.

Checking the LCM we find, in case (a), $\operatorname{lcm}(F_{21}, L_{24}) = 2\cdot 13\cdot 421\cdot 47\cdot 1103 = 567451586$ and, in case (b), $\operatorname{lcm}(F_{25}, L_{22}) = 2971215075$. Case (b) does not give the desired LCM.

Hence, the answer is F_{21} and L_{24}.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, Daina Krigens, Carl Libis, H.-J. Seiffert, Indulis Strazdins, Andràs Szilàrd. and the proposer.

Matrix Lucas Sequence

B-863 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 36, no. 5, November 1998)***

Let

$$A=\begin{pmatrix}-9 & 1\\ -89 & 10\end{pmatrix},\quad B=\begin{pmatrix}-10 & 1\\ -109 & 11\end{pmatrix},\quad C=\begin{pmatrix}-7 & 5\\ -11 & 8\end{pmatrix},\quad \text{and}\quad D=\begin{pmatrix}-4 & 19\\ -1 & 5\end{pmatrix},$$

and let n be a positive integer. Simplify $30A^n - 24B^n - 5C^n + D^n$.

Solution by Hans Kappus, Rodersdorf, Switzerland

It is easily checked that the matrix equation $X^2 = X + I$, where I is the identity matrix, is true for $X = A, B, C,$ and D. Hence, the matrices $M_n = 30A^n - 24B^n - 5C^n + D^n$, $n = 0, 1, 2, \ldots$, satisfy the recurrence $M_{n+2} = M_{n+1} + M_n$. Furthermore, $M_0 = 2I$ and $M_1 = I$. Therefore,

$$M_n = L_n I = \begin{pmatrix}L_n & 0\\ 0 & L_n\end{pmatrix},\quad n = 0, 1, 2, \ldots.$$

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Carl Libis, Maitland A. Rose, H.-J. Seiffert, Andràs Szilàrd, and the proposer.

Confound Those Congruences

B-864 ***Proposed by Stanley Rabinowitz, Westford, MA***
(Vol. 36, no. 5, November 1998)

The sequence $\langle Q_n \rangle$ is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n > 1$ with initial conditions $Q_0 = 2$ and $Q_1 = 2$.

(a) Show that $Q_{7n} \equiv L_n \pmod{159}$ for all n.
(b) Find an integer $m > 1$ such that $Q_{11n} \equiv L_n \pmod{m}$ for all n.
(c) Find an integer a such that $Q_{an} \equiv L_n \pmod{31}$ for all n.
(d) Show that there is no integer a such that $Q_{an} \equiv L_n \pmod{7}$ for all n.
(e) Extra credit: Find an integer $m > 1$ such that $Q_{19n} \equiv L_n \pmod{m}$ for all n.

Solution by David M. Bloom, Brooklyn College of CUNY, Brooklyn , NY

For $k \geq 0$, we have

$$Q_{n+k} + (-1)^k Q_{n-k} = Q_n Q_k \tag{1}$$

by induction on k. (The cases $k = 0$ and $k = 1$ are easy.) Next, we show the following:

$$\text{If } a \text{ is odd and } m \mid (Q_a - 1), \text{ then } Q_{an} \equiv L_n \pmod{m} \text{ for all } n. \tag{2}$$

Indeed, (2) holds for $n = 0$ trivially and for $n = 1$ by hypothesis on m, and, if true for $n = j - 1$, then

$$\begin{aligned} Q_{a(j+1)} &= Q_{a(j-1)} + Q_{aj} Q_a && \text{[by (1)]} \\ &\equiv Q_{a(j-1)} + Q_{aj} \pmod{m} && (\text{since } Q_a = 1) \\ &\equiv L_{j-1} + L_j \pmod{m} && (\text{by the induction hypothesis}) \\ &\equiv L_{j+1} \pmod{m}, \end{aligned}$$

so that (2) holds for $n = j + 1$. Hence, (2) holds for all n by induction.

Part (a) of the problem now follows from (2) with $a = 7$ since $Q_7 - 1 = 477$ is divisible by 159.

Part (b) holds with $m = 13$ since 13 divides $Q_{11} - 1$.

Part (c) holds with $a = 17$ since 31 divides $Q_{17} - 1$.

Part (e) holds with $m = Q_{19} - 1 = 18738637$.

Finally, if $Q_{an} \equiv L_n \pmod{7}$ for all n, then, in particular, $Q_a \equiv L_1 = 1 \pmod{7}$. However, this is impossible since we have (mod 7)

$$(Q_0, Q_1, Q_2, \ldots) \equiv (2, 2, 6, 0, 6, 5, 2, 2, \ldots),$$

which clearly repeats with period 6 and never assumes the value 1. Thus, part (d) is proved.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, H.-J. Seiffert, Andràs Szilàrd, and the proposer.

Belated Acknowledgment: Brian Beasley was inadvertently omitted as a solver of Problems B-854, B-855, and B-857.

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