ON THE INTEGERS OF THE FORM $n(n-1)-1$

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1. AIM OF THE NOTE

The principal aim of this short note is to put into evidence a quite interesting property of the integers M_n given by the left-hand side of the Fibonacci characteristic equation

$$x^2 - x - 1 = 0 \tag{1.1}$$

taken at integers. More precisely, let us define the odd numbers M_n as

$$M_n := n(n-1) - 1 = n^2 - n - 1 \quad (n \geq 2 \text{ an integer}). \tag{1.2}$$

After establishing two marginal properties of the numbers M_n, we prove their main property: namely, for $n \geq 3$, their canonical decomposition does not contain primes of the form $10h \pm 3$. A brief discussion on which numbers M_n are also Fibonacci or Lucas numbers concludes our note.

2. MARGINAL PROPERTIES OF THE NUMBERS M_n

Proposition 1:

$$M_n \equiv \begin{cases} 1 \pmod{10} \\ 5 \pmod{10} \\ 9 \pmod{10} \end{cases} \text{ if } \quad n \equiv \begin{cases} 2, 4, 7, \text{ or } 9 & \pmod{10} \\ 3 \text{ or } 8 & \pmod{10} \\ 0, 1, 5, \text{ or } 6 & \pmod{10}. \end{cases} \tag{2.1}$$

Proposition 1 can be proved by simply computing (1.2) modulo 10.

Proposition 2: For $n \geq 2$, M_n is not divisible by 25.

Proof: From (2.1), we see that, for M_n to be divisible by 5, one must have $n = 5h+3$ ($h = 0$, 1, 2, ...). Consequently, from (1.2), we have $M_{5h+3} = 25h^2 + 25h + 5 \equiv 5 \pmod{25}$.

3. MAIN RESULT

Proposition 3: For $n \geq 3$, the canonical decomposition of M_n has the form

$$M_n = 5^t \prod_{k=1}^{\infty} p_k^{s_k}, \tag{3.1}$$

where t is either 0 or 1 and p_k is a prime of the form $10h \pm 1$ with s_k a nonnegative integer. In particular, the canonical decomposition of M_n does not contain primes of the form $10h \pm 3$.

Remark: If M_n is a prime, then the statement of Proposition 3 and that of Proposition 1 coincide.

Proof of Proposition 3: From (1.2) and Proposition 2, it is sufficient to prove that the incongruence

$$n^2 - n - 1 \not\equiv 0 \pmod{10h \pm 3} \quad (10h \pm 3 \text{ a prime}) \tag{3.2}$$

holds true for all n. Let D $(=5)$ be the discriminant of the equation $x^2-x-1=0$. In [3, p. 223] it is shown how the solution of the congruence $x^2-x-1\equiv 0 \pmod q$ (q a prime) is given by the solution of the congruence $z^2\equiv D \pmod q$. It follows that a sufficient condition for the incongruence (3.2) to be satisfied is that the congruence $z^2\equiv 5 \pmod{10h\pm 3}$ has no solutions. In other words, denoting by (m/p) (p an odd prime, m an integer not divisible by p) the Legendre symbol, to prove (3.2) we have to prove that

$$(5/10h\pm 3)=-1. \tag{3.3}$$

To obtain (3.3), first use the reciprocity law for (m/p) (e.g., see [3, p. 322]), thus getting

$$\begin{cases}(5/10h+3)(10h+3/5)=(-1)^{(5-1)/2\cdot(10h+2)/2}=(-1)^{10h+2}\\(5/10h-3)(10h-3/5)=(-1)^{(5-1)/2\cdot(10h-4)/2}=(-1)^{10h-4}\end{cases}$$

whence

$$(5/10h\pm 3)(10h\pm 3/5)=1. \tag{3.4}$$

Then, on using the property $(m/p)\equiv m^{(p-1)/2} \pmod p$ (see [3, p. 315]), write

$$\begin{aligned}(10h\pm 3/5)&\equiv(10h\pm 3)^{(5-1)/2} \pmod 5\\&\equiv(\pm 3)^2\equiv 9\equiv -1 \pmod 5\end{aligned}$$

whence

$$(10h\pm 3/5)=-1. \tag{3.5}$$

The validity of (3.3) follows necessarily from (3.5) and (3.4). □

An Observation: At first sight, we were amazed at the relatively large number of prime M_n (cf. Sequences 179 and 1558 of [4]): we found 48 of them for $3\le n\le 100$ and 311 of them for $3\le n\le 1000$, whereas it can be seen readily [2] that the expected number of primes in a set of 1000 odd numbers randomly chosen in $[3, 10^6]$ is 157. Actually, the fact that there are so many prime M_n is not surprising, for we know, from Proposition 3, that M_n is not divisible by 3 (or by 7), and that most of the composite numbers are.

4. A QUESTION ABOUT THE NUMBERS M_n

We observed that

$$\begin{aligned}&M_2=F_1=F_2=L_1, && M_6=L_7,\\&M_3=F_5, && M_8=F_{10},\\&M_4=L_5, && M_{10}=F_{11}.\end{aligned} \tag{4.1}$$

A computer experiment allows us to ascertain that, for $11\le n\le 10^{10}$, no numbers M_n are Fibonacci or Lucas numbers. This experiment was carried out by seeking values of k for which the discriminant $4F_k+5$ (resp. $4L_k+5$) of the equation $n^2-n-1=F_k$ (resp. $=L_k$) is a perfect square.

Question: Do there exist numbers M_n that are Fibonacci or Lucas numbers besides those given in (4.1)?

Remark: By virtue of the identity $4L_{2k}+(-1)^k 8=(2L_k)^2$ (see identities I_{15} and I_{18} of [1]), it is not hard to prove that M_n cannot equal an even-subscripted Lucas number.

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