LUCAS SEQUENCES AND FUNCTIONS OF A 4-BY-4 MATRIX

R. S. Melham

School of Mathematical Sciences, University of Technology, Sydney,
PO Box 123, Broadway, NSW 2007 Australia

(Submitted November 1997-Final Revision April 1998)

1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases} \tag{1.1}$$

where p and q are real numbers with $q(p^2-4q) \neq 0$. These sequences were studied originally by Lucas [6], and have subsequently been the subject of much attention.

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\alpha = \frac{p + \sqrt{p^2-4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2-4q}}{2} \tag{1.3}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an n^{th} root of unity for any n.

A well-known relationship between U_n and V_n is

$$V_n = U_{n+1} - qU_{n-1}, \tag{1.4}$$

which we use subsequently.

Recently, Melham [7] considered functions of a 3-by-3 matrix and obtained infinite sums involving squares of terms from the sequences (1.1). Here, using a similarly defined 4-by-4 matrix, we obtain new infinite sums involving cubes, and other terms of degree three, from the sequences (1.1). For example, closed expressions for

$$\sum_{n=0}^{\infty} \frac{U_n^3}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{U_n^2 U_{n+1}}{n!}$$

arise as special cases of results in Section 3 [see (3.4) and (3.5)]. Since the above mentioned paper of Melham contains a comprehensive list of references, we have chosen not to repeat them here.

Unfortunately, one of the matrices which we need to record does not fit comfortably on a standard page. We overcome this difficulty by simply listing elements in a table. Following convention, the (i, j) element is the element in the i^{th} row and j^{th} column.

2. THE MATRIX $A_{k,x}$

By lengthy but straightforward induction on n, it can be shown that the 4-by-4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -q^3 \\ 0 & 0 & q^2 & 3pq^2 \\ 0 & -q & -2pq & -3p^2q \\ 1 & p & p^2 & p^3 \end{pmatrix} \tag{2.1}$$

is such that, for nonnegative integers n, A^n is as follows:

$$\begin{pmatrix} -q^3U_{n-1}^3 & -q^3U_{n-1}^2U_n & -q^3U_{n-1}U_n^2 & -q^3U_n^3 \\ 3q^2U_{n-1}^2U_n & q^2(2U_n^2U_{n-1}+U_{n+1}U_{n-1}^2) & q^2(U_n^3+2U_{n-1}U_nU_{n+1}) & 3q^2U_n^2U_{n+1} \\ -3qU_{n-1}U_n^2 & -q(U_n^3+2U_{n-1}U_nU_{n+1}) & -q(2U_n^2U_{n+1}+U_{n-1}U_{n+1}^2) & -3qU_nU_{n+1}^2 \\ U_n^3 & U_n^2U_{n+1} & U_nU_{n+1}^2 & U_{n+1}^3 \end{pmatrix}.$$

To complete the proof by induction, we make repeated use of the recurrence for $\{U_n\}$. For example, performing the inductive step for the (2, 2) position, we have

$$\begin{aligned} & -q^3(U_n^3+2U_{n-1}U_nU_{n+1})+3pq^2U_n^2U_{n+1} \\ & = q^2U_n[U_n(-qU_n)+2U_{n+1}(-qU_{n-1})+3pU_nU_{n+1}] \\ & = q^2U_n[U_n(U_{n+2}-pU_{n+1})+2U_{n+1}(U_{n+1}-pU_n)+3pU_nU_{n+1}] \\ & = q^2U_n[2U_{n+1}^2+U_nU_{n+2}] \\ & = q^2[2U_{n+1}^2U_n+U_{n+2}U_n^2], \text{ which is the required expression.} \end{aligned}$$

When $p=1$ and $q=-1$, the matrix A becomes

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

which is a 4-by-4 *Fibonacci matrix*. Other 4-by-4 Fibonacci matrices have been studied, for example, in [3] and [4].

The characteristic equation of A is

$$\lambda^4 - p(p^2-2q)\lambda^3 + q(p^2-2q)(p^2-q)\lambda^2 - pq^3(p^2-2q)\lambda + q^6 = 0.$$

Since $p=\alpha+\beta$ and $q=\alpha\beta$, it is readily verified that α^3, $\alpha^2\beta$, $\alpha\beta^2$, and β^3 are the eigenvalues λ_j $(j=1,2,3,4)$ of A. These eigenvalues are nonzero and distinct because of our assumptions in Section 1.

Associated with A, we define the matrix $A_{k,x}$ by

$$A_{k,x} = xA^k, \tag{2.2}$$

where x is an arbitrary real number and k is a nonnegative integer. From the definition of an eigenvalue, it follows immediately that $x\alpha^{3k}$, $x\alpha^{2k}\beta^k$, $x\alpha^k\beta^{2k}$, and $x\beta^{3k}$ are the eigenvalues of $A_{k,x}$. Again, they are nonzero and distinct.

3. THE MAIN RESULTS

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series whose domain of convergence includes the eigenvalues of $A_{k,x}$. Then we have, from (2.2),

$$f(A_{k,x})=\sum_{n=0}^{\infty}a_nA_{k,x}^n=\sum_{n=0}^{\infty}a_nx^nA^{kn}. \tag{3.1}$$

The final sum in (3.1) can be expressed as a 4-by-4 matrix whose entries we record in the following table.

(i,j)	(i,j) element of $f(A_{k,x})$
(1, 1)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^3$
(1, 2)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^2U_{kn}$
(1, 3)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn-1}U_{kn}^2$
(1, 4)	$-q^3\sum_{n=0}^{\infty}a_nx^nU_{kn}^3$
(2, 1)	$3q^2\sum_{n=0}^{\infty}a_nx^nU_{kn-1}^2U_{kn}$
(2, 2)	$q^2\sum_{n=0}^{\infty}a_nx^n(2U_{kn}^2U_{kn-1}+U_{kn+1}U_{kn-1}^2)$
(2, 3)	$q^2\sum_{n=0}^{\infty}a_nx^n(U_{kn}^3+2U_{kn-1}U_{kn}U_{kn+1})$
(2, 4)	$3q^2\sum_{n=0}^{\infty}a_nx^nU_{kn}^2U_{kn+1}$
(3, 1)	$-3q\sum_{n=0}^{\infty}a_nx^nU_{kn-1}U_{kn}^2$
(3, 2)	$-q\sum_{n=0}^{\infty}a_nx^n(U_{kn}^3+2U_{kn-1}U_{kn}U_{kn+1})$
(3, 3)	$-q\sum_{n=0}^{\infty}a_nx^n(2U_{kn}^2U_{kn+1}+U_{kn-1}U_{kn+1}^2)$
(3, 4)	$-3q\sum_{n=0}^{\infty}a_nx^nU_{kn}U_{kn+1}^2$
(4, 1)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}^3$
(4, 2)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}^2U_{kn+1}$
(4, 3)	$\sum_{n=0}^{\infty}a_nx^nU_{kn}U_{kn+1}^2$
(4, 4)	$\sum_{n=0}^{\infty}a_nx^nU_{kn+1}^3$

On the other hand, from the theory of functions of matrices ([2] and [5]), it is known that

$$f(A_{k,x}) = c_0 I + c_1 A_{k,x} + c_2 A_{k,x}^2 + c_3 A_{k,x}^3, \tag{3.2}$$

where I is the 4-by-4 identity matrix, and where c_0, c_1, c_2, and c_3 can be obtained by solving the system

$$\begin{cases} c_0 + c_1 x\alpha^{3k} + c_2 x^2\alpha^{6k} + c_3 x^3 \alpha^{9k} = f(x\lambda_1^k) = f(x\alpha^{3k}), \\ c_0 + c_1 x\alpha^{2k}\beta^k + c_2 x^2 \alpha^{4k}\beta^{2k} + c_3 x^3 \alpha^{6k}\beta^{3k} = f(x\lambda_2^k) = f(x\alpha^{2k}\beta^k), \\ c_0 + c_1 x\alpha^k\beta^{2k} + c_2 x^2\alpha^{2k}\beta^{4k} + c_3 x^3\alpha^{3k}\beta^{6k} = f(x\lambda_3^k) = f(x\alpha^k\beta^{2k}), \\ c_0 + c_1 x\beta^{3k} + c_2 x^2\beta^{6k} + c_3 x^3\beta^{9k} = f(x\lambda_4^k) = f(x\beta^{3k}). \end{cases}$$

With the use of Cramer's rule, and making use of the Binet form for U_n, we obtain, after much tedious algebra,

$$c_0 = \frac{-f(x\alpha^{3k})\beta^{6k}}{U_k U_{2k} U_{3k}(\alpha-\beta)^3} + \frac{f(x\alpha^{2k}\beta^k)\alpha^k\beta^{3k}}{U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\alpha^k\beta^{2k})\alpha^{3k}\beta^k}{U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\beta^{3k})\alpha^{6k}}{U_k U_{2k} U_{3k}(\alpha-\beta)^3},$$

$$c_1 = \frac{f(x\alpha^{3k})\beta^{3k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x\alpha^{2k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} - \frac{f(x\alpha^{2k}\beta^k)(\alpha^{3k}+\beta^{3k}+\alpha^{2k}\beta^k)}{x\alpha^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\alpha^k\beta^{2k})(\alpha^{3k}+\beta^{3k}+\alpha^k\beta^{2k})}{x\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\beta^{3k})\alpha^{3k}(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x\beta^{2k}U_k U_{2k} U_{3k}(\alpha-\beta)^3},$$

$$c_2 = \frac{-f(x\alpha^{3k})\beta^k(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x^2\alpha^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} + \frac{f(x\alpha^{2k}\beta^k)(\alpha^{3k}+\beta^{3k}+\alpha^k\beta^{2k})}{x^2\alpha^{3k}\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\alpha^k\beta^{2k})(\alpha^{3k}+\beta^{3k}+\alpha^{2k}\beta^k)}{x^2\alpha^{2k}\beta^{3k}U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\beta^{3k})\alpha^k(\alpha^{2k}+\beta^{2k}+\alpha^k\beta^k)}{x^2\beta^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3},$$

$$c_3 = \frac{f(x\alpha^{3k})}{x^3\alpha^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3} - \frac{f(x\alpha^{2k}\beta^k)}{x^3\alpha^{3k}\beta^{2k}U_k^2 U_{2k}(\alpha-\beta)^3} + \frac{f(x\alpha^k\beta^{2k})}{x^3\alpha^{2k}\beta^{3k}U_k^2 U_{2k}(\alpha-\beta)^3} - \frac{f(x\beta^{3k})}{x^3\beta^{3k}U_k U_{2k} U_{3k}(\alpha-\beta)^3}.$$

The symmetry in these expressions emerges if we compare the coefficients of $f(x\alpha^{3k})$ and $f(x\beta^{3k})$ and the coefficients of $f(x\alpha^{2k}\beta^k)$ and $f(x\alpha^k\beta^{2k})$.

Now, if we consider (3.1) and (3.2) and the expressions for the entries of A^n, and equate entries in the (4, 1) position, we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = c_1 x U_k^3 + c_2 x^2 U_{2k}^3 + c_3 x^3 U_{3k}^3. \tag{3.3}$$

Finally, with the values of c_1, c_2, and c_3 obtained above, we obtain, with much needed help from the software package "Mathematica":

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^3 = \frac{f(x\alpha^{3k}) - 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) - f(x\beta^{3k})}{(\alpha-\beta)^3}. \tag{3.4}$$

In precisely the same manner, we equate appropriate entries in (3.1) and (3.2) to obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 U_{kn+1} = \frac{\alpha f(x\alpha^{3k}) - (2\alpha+\beta)f(x\alpha^{2k}\beta^k) + (\alpha+2\beta)f(x\alpha^k\beta^{2k}) - \beta f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1}^2 = \frac{\alpha^2 f(x\alpha^{3k}) - (\alpha^2+2\alpha\beta)f(x\alpha^{2k}\beta^k) + (\beta^2+2\alpha\beta)f(x\alpha^k\beta^{2k}) - \beta^2 f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1}^3 = \frac{\alpha^3 f(x\alpha^{3k}) - 3\alpha^2\beta f(x\alpha^{2k}\beta^k) + 3\alpha\beta^2 f(x\alpha^k\beta^{2k}) - \beta^3 f(x\beta^{3k})}{(\alpha-\beta)^3}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn}^2 = \frac{\beta f(x\alpha^{3k}) - (\alpha+2\beta)f(x\alpha^{2k}\beta^k) + (2\alpha+\beta)f(x\alpha^k\beta^{2k}) - \alpha f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} a_n x^n (U_{kn}^3 + 2U_{kn-1}U_{kn}U_{kn+1}) = \frac{3\alpha\beta(f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha+2\beta)(2\alpha+\beta)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha\beta(\alpha-\beta)^3}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn+1} + U_{kn-1}U_{kn+1}^2) = \frac{3\alpha^2\beta f(x\alpha^{3k}) - \alpha(\alpha+2\beta)^2 f(x\alpha^{2k}\beta^k) + \beta(2\alpha+\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha\beta^2 f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 U_{kn} = \frac{\beta^2 f(x\alpha^{3k}) - \beta(2\alpha+\beta)f(x\alpha^{2k}\beta^k) + \alpha(\alpha+2\beta)f(x\alpha^k\beta^{2k}) - \alpha^2 f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} a_n x^n (2U_{kn}^2 U_{kn-1} + U_{kn+1}U_{kn-1}^2) = \frac{3\alpha\beta^2 f(x\alpha^{3k}) - \beta(2\alpha+\beta)^2 f(x\alpha^{2k}\beta^k) + \alpha(\alpha+2\beta)^2 f(x\alpha^k\beta^{2k}) - 3\alpha^2\beta f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^3 = \frac{\beta^3 f(x\alpha^{3k}) - 3\alpha\beta^2 f(x\alpha^{2k}\beta^k) + 3\alpha^2\beta f(x\alpha^k\beta^{2k}) - \alpha^3 f(x\beta^{3k})}{\alpha^3\beta^3(\alpha-\beta)^3}. \tag{3.13}$$

From (3.4) and (3.9), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn} U_{kn+1} = \frac{\alpha\beta(f(x\alpha^{3k}) - f(x\beta^{3k})) - (\alpha^2+\alpha\beta+\beta^2)(f(x\alpha^{2k}\beta^k) - f(x\alpha^k\beta^{2k}))}{\alpha\beta(\alpha-\beta)^3}. \tag{3.14}$$

Similarly, (3.5) and (3.10) and then (3.8) and (3.12) yield, respectively,

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn+1}^2 = \frac{\alpha^2\beta f(x\alpha^{3k}) - \alpha(\alpha^2+2\beta^2) f(x\alpha^{2k}\beta^k) + \beta(2\alpha^2+\beta^2) f(x\alpha^k\beta^{2k}) - \alpha\beta^2 f(x\beta^{3k})}{\alpha\beta(\alpha-\beta)^3}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1} U_{kn-1}^2 = \frac{\alpha\beta^2 f(x\alpha^{3k}) - \beta(2\alpha^2+\beta^2) f(x\alpha^{2k}\beta^k) + \alpha(\alpha^2+2\beta^2) f(x\alpha^k\beta^{2k}) - \alpha^2\beta f(x\beta^{3k})}{\alpha^2\beta^2(\alpha-\beta)^3}. \tag{3.16}$$

Finally, from (1.2), we have $V_{kn}^3 = U_{kn+1}^3 - 3qU_{kn+1}^2 U_{kn-1} + 3q^2 U_{kn+1} U_{kn-1}^2 - q^3 U_{kn-1}^3$. This, together with (3.7), (3.13), (3.15), and (3.16), yields

$$\sum_{n=0}^{\infty} a_n x^n V_{kn}^3 = f(x\alpha^{3k}) + 3f(x\alpha^{2k}\beta^k) + 3f(x\alpha^k\beta^{2k}) + f(x\beta^{3k}) \tag{3.17}$$

after some tedious manipulation involving the use of the equality $\alpha\beta = q$.

4. APPLICATIONS

We now specialize (3.4) and (3.17) to the Chebyshev polynomials to obtain some attractive sums involving third powers of the sine and cosine functions.

Let $\{T_n(t)\}_{n=0}^{\infty}$ and $\{S_n(t)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{aligned} S_n(t) &= \frac{\sin n\theta}{\sin\theta} \\ T_n(t) &= \cos n\theta \end{aligned}\right\}, \quad t = \cos\theta,\ n \ge 0.$$

Indeed, $\{S_n(t)\}_{n=0}^{\infty}$ and $\{2T_n(t)\}_{n=0}^{\infty}$ are the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively, generated by (1.1), where $p = 2\cos\theta$ and $q = 1$. Thus,

$$\alpha = \cos\theta + i\sin\theta = e^{i\theta} \quad \text{and} \quad \beta = \cos\theta - i\sin\theta = e^{-i\theta},$$

which are obtained from (1.3). Further information about Chebyshev polynomials can be found, for example, in [1].

We use the following well-known power series, each of which has the complex plane as its domain of convergence:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \tag{4.1}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \tag{4.2}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \tag{4.3}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{4.4}$$

Now, in (3.4), taking $U_n = \sin n\theta / \sin\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain , after replacing all occurrences of $k\theta$ by ϕ,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sin^3(2n+1)\phi}{(2n+1)!} = \frac{3\cos(x\cos\phi)\sinh(x\sin\phi) - \cos(x\cos 3\phi)\sinh(x\sin 3\phi)}{4}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \sin^3 2n\phi}{(2n)!} = \frac{-3\sin(x\cos\phi)\sinh(x\sin\phi) + \sin(x\cos 3\phi)\sinh(x\sin 3\phi)}{4}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \sin^3(2n+1)\phi}{(2n+1)!} = \frac{3\cosh(x\cos\phi)\sin(x\sin\phi) - \cosh(x\cos 3\phi)\sin(x\sin 3\phi)}{4}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \sin^3 2n\phi}{(2n)!} = \frac{3\sinh(x\cos\phi)\sin(x\sin\phi) - \sinh(x\cos 3\phi)\sin(x\sin 3\phi)}{4}. \tag{4.8}$$

Similarly, in (3.17), taking $V_n = 2\cos n\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, respectively,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cos^3(2n+1)\phi}{(2n+1)!} = \frac{3\sin(x\cos\phi)\cosh(x\sin\phi) + \sin(x\cos 3\phi)\cosh(x\sin 3\phi)}{4}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cos^3 2n\phi}{(2n)!} = \frac{3\cos(x\cos\phi)\cosh(x\sin\phi) + \cos(x\cos 3\phi)\cosh(x\sin 3\phi)}{4}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \cos^3(2n+1)\phi}{(2n+1)!} = \frac{3\sinh(x\cos\phi)\cos(x\sin\phi) + \sinh(x\cos 3\phi)\cos(x\sin 3\phi)}{4}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \cos^3 2n\phi}{(2n)!} = \frac{3\cosh(x\cos\phi)\cos(x\sin\phi) + \cosh(x\cos 3\phi)\cos(x\sin 3\phi)}{4}. \tag{4.12}$$

Finally, we mention that much of the tedious algebra in this paper was accomplished with the help of "Mathematica".

ACKNOWLEDGMENT

The author gratefully acknowledges the input of an anonymous referee, whose suggestions have improved the presentation of this paper.

REFERENCES

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. R. Bellman. *Introduction to Matrix Analysis*. New York: McGraw-Hill, 1970.
3. O. Brugia & P. Filipponi. "Functions of the Kronecker Square of the Matrix Q." In *Applications of Fibonacci Numbers* **2**:69-76. Ed. A. N. Philippou et al. Dordrecht: Kluwer, 1988.
4. P. Filipponi. "A Family of 4-by-4 Fibonacci Matrices." *The Fibonacci Quarterly* **35.4** (1997):300-08.
5. F. R. Gantmacher. *The Theory of Matrices*. New York: Chelsea, 1960.
6. E. Lucas. "Théorie des Fonctions Numériques Simplement Periodiques." *Amer. J. Math.* **1** (1878):184-240, 289-321.
7. R. S. Melham. "Lucas Sequences and Functions of a 3-by-3 Matrix." *The Fibonacci Quarterly* **37.2** (1999):111-16.

AMS Classification Numbers: 11B39, 15A36, 30B10

❖❖❖