GENERALIZED FIBONACCI SEQUENCES AND A GENERALIZATION OF THE Q-MATRIX*

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1. INTRODUCTION

In the notation of Horadam [7], let $W_n = W_n(a, b; p, q)$, where

$$\begin{aligned} &W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2) \\ &W_0 = a, \; W_1 = b. \end{aligned} \tag{1}$$

If α and β, assumed distinct, are the roots of

$$\lambda^2 - p\lambda + q = 0, \tag{2}$$

we have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{3}$$

in which $A = b - a\beta$ and $B = b - a\alpha$.

The n^{th} terms of the well-known Fibonacci and Lucas sequences are then $F_n = W_n(0, 1; 1, -1)$ and $L_n = W_n(2, 1; 1, -1)$.

We also write

$$U_n = W_n(0, 1; p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = W_n(2, p; p, q) = \alpha^n + \beta^n.$$

Throughout this paper, d is a natural number.

Define the Aitken transformation (see [1]) by

$$A(x, x', x'') = \frac{xx'' - x'^2}{x - 2x' + x''}. \tag{4}$$

In 1984, Phillips discovered the following relation between Fibonacci numbers and the Aitken transformation: $A(r_{n-t}, r_n, r_{n+t}) = r_{2n}$, where $r_n = F_{n+1} / F_n$ and $t < n$ is a positive integer, and an account of this work is also given by Vajda in [16]. Later, some articles discussed and extended Phillips' results. For example, McCabe and Phillips [11], Muskat [14], Jamieson [10]. More recently, Zhang [17] defined a generalized Fibonacci sequence as

$$W_{n,d}^{(k)} = W_{n,d}^{(k)}(a, b; p, q) = \frac{A^k\alpha^{nk+d} - B^k\beta^{nk+d}}{\alpha - \beta} \tag{5}$$

and obtained

$$A(R_{n-t}^{(k)}, R_n^{(k)}, R_{n+t}^{(k)}) = R_n^{(2k)}, \tag{6}$$

where $R_n^{(k)} = W_{n,d}^{(k)} / W_{n,0}^{(k)}$. This work generalizes the results of [11], [14], and [10].

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Applying the definition of $W_{n,d}^{(k)}$, we can easily prove that $W_{n,d}^{(k)}$ satisfies the following recurrence relation:

$$W_{n+1,d}^{(k)} = (\alpha^k + \beta^k) W_{n,d}^{(k)} - \alpha^k \beta^k W_{n-1,d}^{(k)}, \tag{7}$$

which has the characteristic equation with roots α^k and β^k.

In this article, Section 2 contains the relation between ratios of $W_{n,d}^{(k)}$ and other transformations and Section 3 gives a generalization of the Q-matrix.

2. THE SECANT, NEWTON-RAPHSON, AND HALLEY TRANSFORMATIONS

If the roots of (2) are real when k tends to infinity, then the sequences of ratios

$$\left\{ R_n^{(k)} = \frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}} \right\}$$

converges to the d^{th} power of a root of (2). In other words, the sequences of ratios $\{R_n^{(k)}\}$ converges to a root of

$$x^2 - (\alpha^d + \beta^d)x + \alpha^d \beta^d = x^2 - V_d x + q^d = 0, \tag{8}$$

namely, $R_n^{(k)} \to \alpha^d$ or β^d as $k \to \infty$.

Define the Secant transformation $S(x, x')$ (see [14]) for equation (8) by

$$S(x, x') = \frac{x(x'^2 - V_d x' + q^d) - x'(x^2 - V_d x + q^d)}{(x'^2 - V_d x' + q^d) - (x^2 - V_d x + q^d)} = \frac{xx' - q^d}{x + x' - V_d}. \tag{9}$$

Define the Newton-Raphson transformation $N(x)$ (see [14]) for equation (8) by

$$N(x) = x - \frac{x^2 - V_d x + q^d}{2x - V_d} = \frac{x^2 - q^d}{2x - V_d}, \tag{10}$$

and the Halley transformation $H(x)$ (see [4]) for equation (8) by

$$H(x) = x - \frac{x^2 - V_d x + q^d}{(2x - V_d) - \frac{x^2 - V_d x + q^d}{2x - V_d}} = \frac{x^3 - 3q^d x + V_d q^d}{3x^2 - 3V_d x + V_d^2 - q^d}. \tag{11}$$

Then we have the following result.

Theorem 1: Let n and m be integers such that $m+n$ is even, and assume that division by zero does not occur. Then:

(i) $S(R_n^{(k)}, R_m^{(k)}) = R_{(m+n)/2}^{(2k)}$, where

$$R_{(m+n)/2}^{(2k)} = \frac{W_{(m+n)/2,d}^{(2k)}}{W_{(m+n)/2,0}^{(2k)}} = \frac{A^{2k}\alpha^{(m+n)k+d} - B^{2k}\beta^{(m+n)k+d}}{A^{2k}\alpha^{(m+n)k} - B^{2k}\beta^{(m+n)k}}; \tag{12}$$

(ii) $N(R_n^{(k)}) = R_n^{(2k)}$; (13)

(iii) $H(R_n^{(k)}) = R_n^{(3k)}$. (14)

Proof: We prove only part (i). The proofs of (ii) and (iii) are similar. Applying the definition and properties—see (3.1)-(3.5) of [17]—of $W_{n,d}^{(k)}$, we have

$$\begin{aligned}
S(R_n^{(k)}, R_m^{(k)}) &= \frac{R_n^{(k)}R_m^{(k)} - q^d}{R_n^{(k)} + R_m^{(k)} - V_d} = \frac{\left(W_{n,d}^{(k)} / W_{n,0}^{(k)}\right)\left(W_{m,d}^{(k)} / W_{m,0}^{(k)}\right) - q^d}{\left(W_{n,d}^{(k)} / W_{n,0}^{(k)}\right) + \left(W_{m,d}^{(k)} / W_{m,0}^{(k)}\right) - V_d} \\
&= \frac{W_{n,d}^{(k)} W_{m,d}^{(k)} - q^d W_{n,0}^{(k)} W_{m,0}^{(k)}}{W_{n,d}^{(k)} W_{m,0}^{(k)} + W_{m,d}^{(k)} W_{n,0}^{(k)} - V_d W_{n,0}^{(k)} W_{m,0}^{(k)}} = \frac{W_{n,d}^{(k)} W_{m,d}^{(k)} - q^d W_{n,0}^{(k)} W_{m,0}^{(k)}}{W_{n,d}^{(k)} W_{m,0}^{(k)} + W_{n,0}^{(k)}\left[W_{m,d}^{(k)} - V_d W_{n,0}^{(k)}\right]} \\
&= \frac{(\alpha^d - \beta^d)(A^{2k}\alpha^{(m+n)k+d} - B^{2k}\beta^{(m+n)k+d}}{(\alpha^d - \beta^d)(A^{2k}\alpha^{(m+n)k} - B^{2k}\beta^{(m+n)k})}.
\end{aligned}$$

This completes the proof of Theorem 1.

We define $\{\Psi_{n,d}^{(k)}\}$, the conjugate sequence of $\{W_{n,d}^{(k)}\}$, by

$$\Psi_{n,d}^{(k)} = \Psi_{n,d}^{(k)}(a, b; p, q) = A^k\alpha^{nk+d} + B^k\beta^{nk+d}. \tag{15}$$

Using (15), it is easy to prove that $\{\Psi_{n,d}^{(k)}\}$ also satisfies the recurrence relation (7). If $\Psi_{n,0}^{(k)} \neq 0$, we use $R_n^{(k)}$ again to denote $\Psi_{n,d}^{(k)} / \Psi_{n,0}^{(k)}$, then this $R_n^{(k)}$ also satisfies the same four relations: (6), (12), (13), and (14).

3. A GENERALIZATION OF THE Q-MATRIX

Before proceeding, we state some results that will be used subsequently. These results can be proved using definitions (5) and (15):

$$\Psi_{n,0}^{(2k)} - 2A^kB^kq^{nk} = \Delta(W_{n,0}^{(k)})^2, \tag{16}$$

$$(W_{n,0}^{(k)})^2 - q^d(W_{n,-d}^{(k)})^2 = U_d W_{n,-d}^{(2k)}, \tag{17}$$

$$(W_{n,d}^{(k)})^2 - q^d(W_{n,0}^{(k)})^2 = U_d W_{n,d}^{(2k)}, \tag{18}$$

$$W_{n,d}^{(k)} W_{n,0}^{(k)} - q^d W_{n,0}^{(k)} W_{n,-d}^{(k)} = U_d W_{n,0}^{(2k)}, \tag{19}$$

$$W_{n,d}^{(m+k)} - A^kB^kq^{nk}W_{n,d}^{(m-k)} = W_{n,0}^{(k)}W_{n,d}^{(m)}, \tag{20}$$

$$W_{n,d}^{(k)} - q^d W_{n,-d}^{(k)} = U_d \Psi_{n,0}^{(k)}, \tag{21}$$

$$(W_{n,0}^{(k)})^2 - W_{n,d}^{(k)} W_{n,-d}^{(k)} = A^kB^kq^{nk-d}U_d^2, \tag{22}$$

where $\Delta = p^2 - 4q$.

Following Hoggatt (see [6]), the Q-matrix is defined by

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Generalizations of the Q-;matrix are to be found in Ivie [9], Filipponi and Horadam [3], Filipponi [2], and Horadam and Filipponi [8]. For a comprehensive history, see Gould [5]. Recently, Melham and Shannon [12], [13], gave the following generalization of the Q-matrix:

$$M=\begin{pmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{pmatrix}.$$

We now give a generalization of the matrix M. Associated with the recurrence relation (7) and with $\{W_{n,d}^{(k)}\}$ and $\{\Psi_{n,d}^{(k)}\}$ as in (5) and (15), respectively, define

$$M_{n,d}^{(k)}=\begin{pmatrix} W_{n,d}^{(k)} & -q^d W_{n,0}^{(k)} \\ W_{n,0}^{(k)} & -q^d W_{n,-d}^{(k)} \end{pmatrix},$$

where k, n, and d are integers.

By induction and making use of (17) and (18), it can be shown that, for all integral n,

$$(M_{n,d}^{(k)})^m=U_d^{m-1}\begin{pmatrix} W_{n,d}^{(mk)} & -q^d W_{n,0}^{(mk)} \\ W_{n,0}^{(mk)} & -q^d W_{n,-d}^{(mk)} \end{pmatrix}.$$

Applying (16)-(20), we obtain the following theorem.

Theorem 2:

$$(M_{n,d}^{(k_1)})^{m_1}(M_{n,d}^{(k_2)})^{m_2}=U_d^{m_1+m_2-1}\begin{pmatrix} W_{n,d}^{(m_1k_1+m_2k_2)} & -q^d W_{n,0}^{(m_1k_1+m_2k_2)} \\ W_{n,0}^{(m_1k_1+m_2k_2)} & -q^d W_{n,-d}^{(m_1k_1+m_2k_2)} \end{pmatrix}. \tag{23}$$

4. A REMARK

In fact, the sequences $W_{n,d}^{(k)}$ and $\Psi_{n,d}^{(k)}$ may be regarded as two double sequences (in n and k, d being a parameter). The interesting properties of the sequences $W_{n,d}^{(k)}$ and $\Psi_{n,d}^{(k)}$ still need further research.

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REFERENCES

1. C. Brezinski. *Accélération de la convergence en analyse numérique*. New York: Springer, 1977.
2. P. Filipponi. "Waring's Formula, the Binomial Formula, and Generalized Fibonacci Matrices." *The Fibonacci Quarterly* **30.3** (1992):225-31.
3. P. Filipponi & A. F. Horadam. "A Matrix Approach to Certain Identities." *The Fibonacci Quarterly* **26.2** (1988):115-26.
4. W. Gander. "On Halley's Iteration Method." *Amer. Math. Monthly* **92** (1985):131-34.
5. H. W. Gould. "A History of the Fibonacci Q-Matrix and a Higher-Dimensional Problem." *The Fibonacci Quarterly* **19.3** (1981):250-57.
6. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, Calif.: The Fibonacci Association, 1979.
7. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.

8. A. F. Horadam & P. Filipponi. "Cholesky Algorithm Matrices of Fibonacci Type and Properties of Generalized Sequences." *The Fibonacci Quarterly* **29.2** (1991):164-73.
9. J. Ivie. "A General Q-Matrix." *The Fibonacci Quarterly* **10.3** (1972):255-61, 264.
10. M. J. Jamieson. "Fibonacci Numbers and Aitken Sequences Revisited." *Amer. Math. Monthly* **97** (1990):829-31.
11. J. H. McCabe & G. M. Phillips. "Aitken Sequences and Generalized Fibonacci Numbers." *Math. Comp.* **45** (1985):553-58.
12. R. S. Melham & A. G. Shannon. "Some Infinite Series Summations Using Power Series Evaluated at a Matrix." *The Fibonacci Quarterly* **33.1** (1995):13-20.
13. R. S. Melham & A. G. Shannon. "Some Summation Identities Using Generalized Q-Matrices." *The Fibonacci Quarterly* **33.1** (1995):64-73.
14. B. Muskat. "Generalized Fibonacci and Lucas Sequences and Rootfinding Methods." *Math. Comp.* **61** (1993):365-72.
15. G. M. Phillips. "Aitken Sequences and Fibonacci Numbers." *Amer. Math. Monthly* **91** (1984):354-57.
16. S. Vajda. *Fibonacci & Lucas Numbers and the Golden Section: Theory and Applications*, pp. 103-04. New York: Ellis Horwood, 1989.
17. Zhizheng Zhang. "A Class of Sequences and the Aitken Transformation." *The Fibonacci Quarterly* **36.1** (1998):68-71.

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