ON THE SEQUENCES $T_n = T_{n-1} + T_{n-2} + hn + k$

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(Submitted March 1998-Final Revision August 1998)

1. INTRODUCTION

A colleague of ours who needed to evaluate the computational complexity of certain algorithms for optimal traffic routing on multi-service networks asked us for a closed-form expression for the sum of the first N terms of the sequences $\{X_n\}$ and $\{Y_n\}$ obeying the second-order nonhomogeneous recurrence relations

$$X_n = X_{n-1} + X_{n-2} + k \quad (k,\, X_0,\text{ and } X_1 \text{ arbitrary}) \tag{1.1}$$

and

$$Y_n = Y_{n-1} + Y_{n-2} + n \quad [Y_0 = 0;\ Y_1 = 1], \tag{1.2}$$

respectively. His request led us to investigate the main properties of the more general sequences $\{T_n(h, k; a, b)\}$ (or simply $\{T_n\}$ if no misunderstanding can arise) defined as

$$T_n = T_{n-1} + T_{n-2} + hn + k \quad [T_0 = a;\ T_1 = b], \tag{1.3}$$

where h, k, a, and b are arbitrary integers. In doing so, beyond answering the question posed by our colleague, we generalize some results established in [1] and [7]. It is worth pointing out that T_n can be expressed either as the third-order inhomogeneous recurrence relation

$$T_n = 2T_{n-1} - T_{n-3} + h \tag{1.4}$$

with initial conditions

$$T_0,\ T_1, \text{ and } T_2 = T_0 + T_1 + 2h + k, \tag{1.4'}$$

or as the fourth-order homogeneous recurrence relation

$$T_n = 3T_{n-1} - 2T_{n-2} - T_{n-3} + T_{n-4} \tag{1.5}$$

with initial conditions given by (1.4') and the additional condition $T_3 = T_0 + 2T_1 + 5h + 2k$.

As usual, throughout the paper, F_n and L_n will denote the n^{th} Fibonacci and Lucas number, respectively.

2. CLOSED-FORM EXPRESSION FOR T_n

The closed-form expression for T_n is, quite obviously, a powerful tool for discovering properties of these numbers. From the definition (1.3), by using standard methods (e.g., see [4]), we found that

$$T_n = AF_{n-1} + BF_n - h(n+3) - k \tag{2.1}$$

where

$$\begin{cases} A = 3h + k + a, \\ B = 4h + k + b. \end{cases} \tag{2.2}$$

The reader can immediately check that (2.1) and (2.2) satisfy both the recurrence and the initial conditions in (1.3). This fact proves the validity of the above expressions.

By using (1.3) or (2.1) and the well-known identity $F_{-n} = (-1)^{n-1}F_n$, the extension of T_n through negative values of the subscript n is readily obtained. Namely, we get

$$T_{-n} = T_n + 2hn - \begin{cases} (5h+k+2b-a)F_n & (n \text{ even}), \\ (3h+k+a)L_n & (n \text{ odd}). \end{cases} \tag{2.3}$$

Observe that the second identity of (2.3) is formally independent of b.

Special cases:

$T_n(0, 0; 0, 1) = F_n$, (2.4)
$T_n(0, 0; 2, 1) = L_n$, (2.5)
$T_n(0, k; a, b) = X_n$ (see (1.1) and [1]), (2.6)
$T_n(1, 0; 0, 1) = Y_n = F_{n+4} - n - 3$ (see (1.2) and Seq. 1053 of [5]), (2.7)
$T_n(h, h; h, h) = h(L_{n+3} - F_{n-2} - n - 4)$, (2.8)
$T_n(0, k; 0, 0) = k(F_{n+1} - 1)$. (2.9)

3. SOME SPECIAL PROPERTIES OF THE SEQUENCES $\{T_n\}$

Here we point out three properties of the sequences $\{T_n\}$ that seem especially interesting to us. Their proofs are given in full detail. Let us state the following.

Proposition 1: For an arbitrarily given integer m, we have

$$T_n(h, k; a, b) = T_{n-m}(h, k+mh;\ T_m(h, k; a, b),\ T_{m+1}(h, k; a, b)). \tag{3.1}$$

Proof: Use (2.1) and (2.2) to rewrite the right-hand side of (3.1) as

$$\begin{aligned}
&[3h+k+mh+AF_{m-1}+BF_m-h(m+3)-k]F_{n-m-1} \\
&\quad +[4h+k+mh+AF_m+BF_{m+1}-h(m+4)-k]F_{n-m}-h(n-m+3)-k-mh \\
&=[AF_{m-1}+BF_m]F_{n-m-1}+[AF_m+BF_{m+1}]F_{n-m}-h(n+3)-k \\
&=A(F_{m-1}F_{n-m-1}+F_mF_{n-m})+B(F_mF_{n-m-1}+F_{m+1}F_{n-m})-h(n+3)-k \\
&=AF_{n-1}+BF_n-h(n+3)-k=T_n(h, k; a, b) \quad [\text{from } \mathrm{I}_{26} \text{ of [3] and (2.1)}]. \ \square
\end{aligned}$$

Proposition 2: For given integers h, k, a, b, h_1, and all n, we have

$$T_n(h, k; a, b) = T_n(h_1, k-(n+3)s; a+ns, b+(n-1)s), \tag{3.2}$$

where $s = h_1 - h$.

Proof: From (2.1), it is patent that identity (3.2) can be obtained by solving the system

$$\begin{cases} A = 3h_1 + k_1 + a_1, \\ B = 4h_1 + k_1 + b_1, \\ h(n+3)+k = h_1(n+3)+k_1. \end{cases} \tag{3.3}$$

Put $h_1 = h+s$ in (3.3) and use the third equation to obtain $k_1 = k-(n+3)s$. Then use (2.2), and replace the above expression for k_1 in the first two equations of (3.3) to get $a_1 = a+ns$ and $b_1 = b+(n-1)s$. $\square$

Finally, we observe [see (2.7)] that there exist values of the parameters $(h, k; a, b)$ for which T_n has the form

$$T_n = F_m + p(n), \tag{3.4}$$

where $p(n)$ is a first-degree polynomial in n. Whereas such a problem is put forward: find conditions on $(h, k; a, b)$ for T_n to have the form (3.4). We give the following proposition.

Proposition 3: For arbitrarily given integers a, b, and s, we have

$$T_n(F_{s-1} + a - b, 3b - L_{s-2} - 4a; a, b) = F_{n+s} - n(F_{s-1} + a - b) - F_s + a. \tag{3.5}$$

Proof: From (2.1), (2.2), and I_{26} of [3], it is evident that (3.4) can be obtained if

$$\begin{cases} A = 3h + k + a = F_s, \\ B = 4h + k + b = F_{s+1}. \end{cases} \tag{3.6}$$

Subtracting the first equation of (3.6) from the second equation, one obtains

$$h = F_{s-1} + a - b, \tag{3.7}$$

and from the first equation,

$$\begin{aligned} k = F_s - 3h - a &= F_s - 3F_{s-1} - 4a + 3b \quad \text{[from (3.7)]} \\ &= 3b - L_{s-2} - 4a. \end{aligned} \tag{3.8}$$

Expressions (3.7) and (3.8) give the left-hand side of (3.5). Its right-hand side can be obtained from (2.1) and (2.2), after some manipulation involving the use of the identities $3F_{s-1} - L_{s-2} = F_s$, $4F_{s-1} - L_{s-2} = F_{s+1}$, and I_{26} of [3]. □

Examples: The right-hand side of (2.7) emerges from the choice $(a, b, s) = (0, 1, 4)$. As a further example, the choice $(a, b, s) = (10, 7, 8)$ yields the numbers $T_n(16, -37; 10, 7) = F_{n+8} - 16n - 11$.

4. BASIC IDENTITIES INVOLVING THE NUMBERS T_n

Here we give a brief account of the basic identities involving the numbers T_n. To save space, the number of detailed proofs will be kept to a minimum (Subsection 4.2).

4.1 Results

Generating function

By using (2.1), (2.2), and [6, p. 53], we get

$$\sum_{n=0}^{\infty} x^n T_n = \frac{(h + k + a - b)x^3 - (2h + k + 3a - 2b)x^2 + (3a - b)x - a}{x^4 - x^3 - 2x^2 + 3x - 1}. \tag{4.1}$$

Observe that, for $T_n \equiv Y_n$ [see (1.2)], the numerator on the right-hand side of (4.1) collapses to $-x$.

Simson formula analog

$$\begin{aligned} \sigma(T_n) &:= T_n^2 - T_{n-1}T_{n+1} \\ &= (-1)^n C + (hn + k)(T_{n-3} + hn + k) - 2h(T_{n-4} + k) - h^2(2n - 3), \end{aligned} \tag{4.2}$$

where

$$C = A^2 + AB - B^2 = 5h(h + k + 2a - b) + k(k + 3a - b) + a^2 + ab - b^2. \tag{4.3}$$

From (4.2), (4.3), and (2.7), we see that

$$\sigma(Y_n) = nF_{n+1} - 2F_n + 1 - (-1)^n. \tag{4.4}$$

Sums and differences

$$T_{n+m} + T_{n-m} = \begin{cases} L_m[T_n + h(n+3) + k] - 2[h(n+3) + k] & (m \text{ even}), \\ F_m[T_n + 2T_{n-1} + h(3n+7) + 3k] - 2[h(n+3) + k] & (m \text{ odd}). \end{cases} \tag{4.5}$$

$$T_{n+m} - T_{n-m} = \begin{cases} F_m[T_n + 2T_{n-1} + h(3n+7) + 3k] - 2hm & (m \text{ even}), \\ L_m[T_n + h(n+3) + k] - 2hm & (m \text{ odd}). \end{cases} \tag{4.6}$$

Duplication formula

For n even (resp. odd), let $m = n$ in the first (resp. second) identity of (4.5) [resp. (4.6)] to obtain

$$T_{2n} = L_n[T_n + h(n+3) + k] - (-1)^n a - \begin{cases} 2[h(n+3) + k] & (n \text{ even}), \\ 2hn & (n \text{ odd}). \end{cases} \tag{4.7}$$

Observe that (4.7) is formally independent of b.

Finite sums

$$S_N(T) := \sum_{n=0}^{N} T_n = T_{N+2} - \frac{(N+1)[h(N+4) + 2k]}{2} - b. \tag{4.8}$$

From (4.8), (1.1), (1.2), (2.1), and (2.2), we obtain the special identities

$$S_N(T \equiv X) = k(F_{N+3} - N - 2) + X_0 F_{N+1} + X_1(F_{N+2} - 1) \tag{4.9}$$

and

$$S_N(T \equiv Y) = F_{N+6} - (N^2 + 7N + 16)/2, \tag{4.10}$$

which answer the questions that gave rise to our study.

Further, we get the identities:

$$\sum_{n=0}^{N} nT_n = NT_{N+2} - T_{N+3} - \frac{h(N^3 + 3N^2 - 7N - 15)}{3} - \frac{k(N^2 - N - 4)}{2} + a + 2b, \tag{4.11}$$

$$\sum_{n=0}^{N} \binom{N}{n} T_n = T_{2N} - h[2^{N-1}(N+6) - 2N - 3] - k(2^N - 1), \tag{4.12}$$

$$\sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} 2^n T_{2n} = T_{3N} - hN, \tag{4.13}$$

the last of which generalizes (19) of [7].

Convolution (for $T_n \equiv Y_n$)

$$\sum_{n=0}^{N} Y_n Y_{N-n} = \frac{NL_{N+8} + F_{N+10}}{5} - L_{N+9} + \frac{N^3 + 18N^2 + 131N}{6} + 65 \tag{4.14}$$

$$= F^{(1)}_{N+8} - 4F_{N+9} + \frac{N^3 + 18N^2 + 131N}{6} + 65, \tag{4.14'}$$

where $F^{(1)}_n$ denotes the n^{th} term of the Fibonacci first derivative sequence [2].

4.2 Proofs

Proof of (4.2) (a sketch): From (2.1) and (2.2), after a good deal of calculation involving the use of some well-known Fibonacci identities ([3], [6]), one gets

$$\begin{aligned}\sigma(T_n) &= (-1)^n C + h^2 + A[(hn+k)F_{n-4} - 2hF_{n-5}] + B[(hn+k)F_{n-3} - 2hF_{n-4}] \\ &= (-1)^n C + h^2 + (hn+k)(AF_{n-4} + BF_{n-3}) - 2h(AF_{n-5} + BF_{n-4}) \\ &= (-1)^n C + (hn+k)T_{n-3} - 2hT_{n-4} + h^2 + (hn+k)^2 - 2h^2(n-1) - 2hk,\end{aligned}$$

whence (4.2) is immediately obtained. □

Proof of (4.6) (for m even): By using (2.1) and (2.2), rewrite the left-hand side of (4.6) as

$$\begin{aligned}&A(F_{n+m-1} - F_{n-m-1}) + B(F_{n+m} - F_{n-m}) - 2hm \\ &= AL_{n-1}F_m + BL_nF_m - 2hm \quad (\text{from } \mathrm{I}_{24} \text{ of } [3]) \\ &= F_m(AL_{n-1} + BL_n) - 2hm = F_m(AF_{n-2} + BF_{n-1} + AF_n + BF_{n+1}) - 2hm \\ &= F_m\{T_{n-1} + T_{n+1} + 2[h(n+3)+k]\} - 2hm \\ &= F_m[T_n + 2T_{n-1} + h(3n+7) + 3k] - 2hm \quad [\text{from } (1.3)]. \ \square\end{aligned}$$

Proofs of (4.8), (4.11), and (4.12): From (4.8) and the recurrence (1.3), write

$$\begin{aligned}S_N(T) &= \sum_{n=0}^{N} T_{n-1} + \sum_{n=0}^{N} T_{n-2} + \sum_{n=0}^{N}(hn+k) \\ &= \sum_{n=-1}^{N-1} T_n + \sum_{n=-2}^{N-2} T_n + H \quad \left(H := \sum_{n=0}^{N}(hn+k)\right) \\ &= S_N(T) - T_N + T_{-1} + S_N(T) - T_N - T_{N-1} + T_{-1} + T_{-2} + H,\end{aligned}$$

whence

$$\begin{aligned}S_N(T) &= 2T_N + T_{N-1} - 2T_{-1} - T_{-2} - h \\ &= T_N + T_{N+1} - h(N+1) - k - (T_{-1} + T_0 - k) - H \quad [\text{from } (1.3)] \\ &= T_{N+2} - h(N+2) - k - h(N+1) - k - (T_1 - h - 2k) - H \\ &= T_{N+2} - 2h(N+1) - b - H.\end{aligned} \tag{4.15}$$

Take the meaning of H into account and use (4.15) to obtain (4.8). The identities (4.11) and (4.12) can be proved by means of a similar technique. □

Proof of (4.13) (Hint):

(i) Identity (4.13) can be proved by means of the technique used by Zhang [7] after replacing (18) of [7] by the identity $T_n = 2T_{n-1} - T_{n-3} + h$, which can be obtained readily from (1.3).

(ii) Alternatively, use (2.1) to rewrite the left-hand side of (4.13) as

$$(-1)^N\left\{A\sum_{n=0}^{N}\binom{N}{n}(-2)^n F_{2n-1} + B\sum_{n=0}^{N}\binom{N}{n}(-2)^n F_{2n} - \sum_{n=0}^{N}\binom{N}{n}(-2)^n[h(2n+3)+k]\right\},$$

and use the Binet form for Fibonacci numbers along with (3.3) and (3.4) of [2]. □

Proofs of (4.14) and (4.14'): First, use (2.7) and the Binet form for Fibonacci numbers to get

$$\begin{aligned} Y_n Y_{N-n} &= (F_{n+4} - n - 3)[F_{N-n+4} + n - (N+3)] \\ &= \frac{L_{N+8} - (-1)^n L_{N-2n}}{5} + nF_{n+4} - (N+3)F_{n+4} - nF_{N-n+4} \\ &\qquad - n^2 + nN - 3F_{N-n+4} + 3(N+3). \end{aligned} \tag{4.16}$$

Then, after denoting the left-hand side of (4.14) by C_N and letting $S[x(n)] := \sum_{n=0}^{N} x(n)$ for notational convenience, use (4.16) to write

$$\begin{aligned} C_N &= \tfrac{1}{5}S[L_{N+8}] - \tfrac{1}{5}S[(-1)^n L_{N-2n}] + S[nF_{n+4}] - (N+3)S[F_{n+4}] \\ &\quad - S[nF_{N-n+4}] - S[n^2] + NS[n] - 3S[F_{N-n+4}] + 3(N+3)S[1] \\ &:= S_1 - S_2 + S_3 - S_4 - S_5 - S_6 + S_7 - S_8 + S_9. \end{aligned} \tag{4.17}$$

By using the Binet forms for Fibonacci and Lucas numbers, the geometric series formula and some well-known identities (I_1 and I_{40} of [3] inclusive), one obtains the partial results,

(i) $S_1 = (N+1)L_{N+8}/5,$

(ii) $S_2 = 2F_{N+1}/5,$

(iii) $S_3 = NF_{N+6} - F_{N+7} + 13,$

(iv) $S_4 = (N+3)(F_{N+6} - 5),$

(v) $S_5 = F_{N+7} - 5(N+4) + 7,$

(vi) $S_6 = N(N+1)(2N+1)/6,$

(vii) $S_7 = N^2(N+1)/2,$

(viii) $S_8 = 3(F_{N+6} - 5),$

(ix) $S_9 = 3(N+3)(N+1),$

among which *(ii)* is quite interesting *per se*. Finally, from (4.17) and *(i)-(ix)*, one finds

$$C_N = \frac{(N+1)L_{N+8} - 2F_{N+1}}{5} - 6F_{N+6} - 2F_{N+7} + \frac{N^3 + 18N^2 + 131N}{6} + 65,$$

from which, by applying properties of Fibonacci-Lucas sequences, (4.14) can be obtained immediately. The right-hand side of (4.14') can be found by using (2.5) of [2] to rewrite the first two addends on the right-hand side of (4.14) as

$$\begin{aligned} F_{N+8}^{(1)} + (F_{N+8} + F_{N+10} - 8L_{N+8} - 5L_{N+9})/5 &= F_{N+8}^{(1)} - 4(L_{N+10} + L_{N+8})/5 \\ &= F_{N+8}^{(1)} - 4F_{N+9} \quad (\text{from } I_9 \text{ of } [3]). \ \square \end{aligned}$$

5. FURTHER WORK

From (4.14), one may observe that

$$Q_n := (nL_{n+8} + F_{n+10})/5 \tag{5.1}$$

is an integer for all n. In fact, it is immediate to check that Q_n obeys the recurrence

$$Q_n = Q_{n-1} + Q_{n-2} + F_{n+7} \quad [Q_0 = 11;\ Q_1 = 33]. \tag{5.2}$$

This fact suggests the idea of studying properties of the more general sequences $\{Q_n(k)\}$, defined by

$$Q_n(k) = Q_{n-1}(k) + Q_{n-2}(k) + F_{n+k} \quad [Q_0(k) = a;\ Q_1(k) = b], \tag{5.3}$$

the elements of which have the closed-form expression

$$Q_n(k) = aF_{n-1} + bF_n + (nL_{n+k+1} - L_{k+2}F_n)/5. \tag{5.4}$$

Much more generally, one might investigate properties of the sequences $\{R_n\}$, defined by

$$R_n = R_{n-1} + R_{n-2} + f_n \quad [R_0 = a;\ R_1 = b], \tag{5.5}$$

where f_n is *any* integer-valued function of n. This study will be the aim of a future paper. For the time being, we confine ourselves to showing a compact form for R_n. Namely, we get

$$R_n = aF_{n-1} + bF_n + \sum_{r=2}^{n} f_r F_{n-r+1}. \tag{5.6}$$

Observe that, as special cases, $R_n = Q_n(k)$ (resp. T_n) for $f_n = F_{n+k}$ (resp. $hn+k$). It can be noted that letting $f_n = hn+k$ in (5.6) yields the expression

$$R_n = T_n = aF_{n-1} + bF_n + h(L_{n+2} - n - 3) + k(F_{n+1} - 1), \tag{5.7}$$

which can be proved easily to be an equivalent form for (2.1). As further special cases, we urge the interested reader to prove that, if $f_n = X^n$, then

$$R_n = (a-1)F_{n-1} + (b - X - 1)F_n + \frac{X^{n+2} - XF_{n+2} - F_{n+1}}{X^2 - X - 1}, \tag{5.8}$$

whereas, if $f_n = F_n$ (resp. L_n) and $(a, b) = (0, 1)$, then

$$R_n = Q_n(0) = \frac{nL_{n+1} + 2F_n}{5} = \frac{(n+1)L_{n+1} - F_{n+1}}{5} = F_{n+1}^{(1)} \tag{5.9}$$

(see (5.4) and [2]), and

$$R_n = nF_{n+1}, \tag{5.10}$$

respectively.

ACKNOWLEDGMENT

This work has been carried out in the framework of an agreement between the Italian PT Administration and the Fondazione Ugo Bordoni. The authors wish to thank the anonymous referee for helpful suggestions. In particular, they are pleased to thank the referee for pointing out recurrence (1.5).

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AMS Classification Numbers: 11B39, 11B83, 05C85