REPRESENTATION GRIDS FOR CERTAIN MORGAN-VOYCE NUMBERS

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1. BACKGROUND

Properties of representation number sequences $\{\mathcal{B}_n\}$, $\{\mathcal{C}_n\}$ associated with the Morgan-Voyce polynomials $B_n(x)$ and the related polynomials $C_n(x)$ were recently investigated in [1]. Hopefully, the notation and references in [1] will be accessible to the reader.

Complementary properties of the number sequences $\{\mathbf{b}_n\}$, $\{\mathbf{c}_n\}$ associated with the Morgan-Voyce polynomials $b_n(x)$ and the related polynomials $c_n(x)$ are now explored.

With $x = 1$ in these just-mentioned polynomials, we define the resulting numbers by

$$b_n = 3b_{n-1} - b_{n-2}, \qquad b_0 = 1,\ b_1 = 1, \tag{1.1}$$

and

$$c_n = 3c_{n-1} - c_{n-2}, \qquad c_0 = -1,\ c_1 = 1. \tag{1.2}$$

Accordingly, these numbers are

$$\begin{array}{rcrrrrrrrrrl} n &=& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots, \\ b_n &=& 1 & 1 & 2 & 5 & 13 & 34 & 89 & 233 & 610 & \dots, \\ c_n &=& -1 & 1 & 4 & 11 & 29 & 76 & 199 & 521 & 1364 & \dots. \end{array} \tag{1.3}$$

Consider now the unit coefficient representation sums for b_n, c_n analogous to those for B_n, C_n [1]. Irrespective of the uniqueness or otherwise of the representations (and of questions of minimality or maximality), we may assert that, for the representation number sequences $\{\mathbf{b}_n\}$, $\{\mathbf{c}_n\}$,

$$\mathbf{b}_n = \sum_{i=1}^{n} b_n = F_{2n} = F_n L_n \tag{1.4}$$

and

$$\mathbf{c}_n = \sum_{i=1}^{n} c_n = L_{2n} - 2 = \begin{cases} L_n^2 & n \text{ odd}, \\ 5F_n^2 & n \text{ even}, \end{cases} \tag{1.5}$$

in terms of the Fibonacci and Lucas numbers F_n, L_n.

Elements of $\{\mathbf{b}_n\}$, $\{\mathbf{c}_n\}$ are thus

$$\begin{array}{rcrrrrrrrrrl} n &=& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots, \\ \mathbf{b}_n &=& 0 & 1 & 3 & 8 & 21 & 55 & 144 & 377 & 987 & \dots, \\ \mathbf{c}_n &=& 0 & 1 & 5 & 16 & 45 & 121 & 320 & 841 & 2205 & \dots. \end{array} \tag{1.6}$$

Why, we may ask, are these numbers worthy of our consideration? Firstly, as mathematical constructs they have an inherent interest to the inquiring mind ("because they are there"!). Secondly, as the theory—necessarily compact—unfolds, they add a little, however modest, to our knowledge of number relationships. Moreover, they complete the theme initiated in [1].

2. PROPERTIES OF $\mathbf{b}_n$, $\mathbf{c}_n$

One may readily establish the fundamental infrastructure of these two number systems, details of which are herewith reported (in pairs, for comparison).

Recurrences:

$$\mathbf{b}_n = 3\mathbf{b}_{n-1} - \mathbf{b}_{n-2}, \tag{2.1}$$

$$\mathbf{c}_n = 3\mathbf{c}_{n-1} - \mathbf{c}_{n-2} + 2. \tag{2.2}$$

Binet forms:

$$\mathbf{b}_n = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \quad (\alpha\beta = 1,\ \alpha\beta = -1), \tag{2.3}$$

$$\mathbf{c}_n = \alpha^{2n} + \beta^{2n} - 2. \tag{2.4}$$

Generating functions:

$$\sum_{i=1}^{\infty} \mathbf{b}_i x^{i-1} = [1 - (3x - x^2)]^{-1}, \tag{2.5}$$

$$\sum_{i=1}^{\infty} \mathbf{c}_i x^{i-1} = (1 + x)[1 - (4x - 4x^2 + x^3)]^{-1}. \tag{2.6}$$

Simson formulas:

$$\mathbf{b}_{n+1}\mathbf{b}_{n-1} - \mathbf{b}_n^2 = -1, \tag{2.7}$$

$$\mathbf{c}_{n+1}\mathbf{c}_{n-1} - \mathbf{c}_n^2 = 1 - 2\mathbf{c}_n. \tag{2.8}$$

Summations:

$$\sum_{i=1}^{n} \mathbf{b}_i = F_{2n+1} - 1, \tag{2.9}$$

$$\sum_{i=1}^{n} \mathbf{c}_i = L_{2n+1} - (2n + 1), \tag{2.10}$$

$$\sum_{i=1}^{n} \mathbf{b}_{2i} = \frac{1}{5}(L_{2n+1}^2 - 5), \tag{2.11}$$

$$\sum_{i=1}^{n} \mathbf{c}_{2i} = F_{4n+2} - (2n + 1), \tag{2.12}$$

$$\sum_{i=1}^{n} \mathbf{b}_{2i-1} = F_{2n}^2, \tag{2.13}$$

$$\sum_{i=1}^{n} \mathbf{c}_{2i-1} = F_{4n} - 2n, \tag{2.14}$$

$$\sum_{i=1}^{n} (-1)^{i+1}\mathbf{b}_i = \frac{1}{5}[1 - (-1)^n L_{2n+1}], \tag{2.15}$$

$$\sum_{i=1}^{n} (-1)^{i+1}\mathbf{c}_i = (-1)^n[1 - F_{2n+1}]. \tag{2.16}$$

Other simple properties:

$$\mathbf{b}_{-n} = -\mathbf{b}_n, \tag{2.17}$$

$$\mathbf{c}_{-n} = \mathbf{c}_n, \tag{2.18}$$

$$\mathbf{b}_n - \mathbf{b}_{n-1} = F_{2n-1}, \tag{2.19}$$

$$\mathbf{c}_n - \mathbf{c}_{n-1} = L_{2n-1}, \tag{2.20}$$

$$\mathbf{b}_n + \mathbf{b}_{n+1} = L_{2n-1} \text{ also}, \tag{2.21}$$

$$\mathbf{c}_n + \mathbf{c}_{n+1} = 5L_{2n-1} - 4, \tag{2.22}$$

$$\mathbf{b}_n - \mathbf{b}_{n-2} = L_{2n-2}, \tag{2.23}$$

$$\mathbf{c}_n - \mathbf{c}_{n-2} = 5F_{2n-2}, \tag{2.24}$$

$$\mathbf{b}_n^2 - \mathbf{b}_{n-1}^2 = F_{4n-2} = F_{2n-1}L_{2n-1}. \tag{2.25}$$

3. THE REPRESENTATION GRIDS

Next, we introduce the concepts

$$\begin{aligned}\mathbf{b}'_n &= \mathbf{b}_{n+1} + \mathbf{b}_{n-1} \\ &= 3\mathbf{b}_n = 3F_{2n} = \mathcal{B}'_n - \mathcal{B}'_{n-1},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\mathbf{c}'_n &= \mathbf{c}_{n+1} + \mathbf{c}_{n-1} \\ &= 3\mathbf{c}_n + 2 = 3L_{2n} - 4 = \mathcal{B}'_n + \mathcal{B}'_{n+1},\end{aligned} \tag{3.2}$$

on invoking [1].

Repeating the summation process developed in [1], i.e., $\mathbf{b}''_n = \mathbf{b}'_{n+1} + \mathbf{b}'_{n-1}$, we eventually arrive at the more general notations

$$\mathbf{b}_n^{(m)} = \mathbf{b}_{n+1}^{(m)} + \mathbf{b}_{n-1}^{(m)} \quad (\mathbf{b}_n^{(0)} = \mathbf{b}_n), \tag{3.3}$$

and

$$\mathbf{c}_n^{(m)} = \mathbf{c}_{n+1}^{(m)} + \mathbf{c}_{n-1}^{(m)} \quad (\mathbf{c}_n^{(0)} = \mathbf{c}_n). \tag{3.4}$$

As in [1], these data can be organized in (representation) *grids* for $\mathbf{b}_n^{(m)}$ and $\mathbf{c}_n^{(m)}$, where m denotes columns and n rows.

Various approaches allow us to validate the properties recorded below, some of which are readily obtainable from the patterns in the rectangular grids, which the reader should construct for visual emphasis and clarification of the theory.

Zero subscripts:

$$\mathbf{b}_0^{(m)} = 0 = \mathcal{B}_0^{(0)}, \tag{3.5}$$

$$\mathbf{c}_0^{(m)} = 2(3^m - 2^m) = 2\mathcal{B}_0^{(m)} = -2\mathcal{C}_0^{(m)} \quad \text{by [1]}. \tag{3.6}$$

Negative subscripts:

$$\mathbf{b}_{-n}^{(m)} = -\mathbf{b}_n^{(m)}, \tag{3.7}$$

$$\mathbf{c}_{-n}^{(m)} = \mathbf{c}_n^{(m)}. \tag{3.8}$$

Recurrences:

$$\text{(columns)}\quad \begin{cases} \mathbf{b}_n^{(m)} = 3\mathbf{b}_{n-1}^{(m)} - \mathbf{b}_{n-2}^{(m)}, & (3.9) \\ \mathbf{c}_n^{(m)} = 3\mathbf{c}_{n-1}^{(m)} - \mathbf{c}_{n-2}^{(m)} + 2^{m+1}, & (3.10) \end{cases}$$

$$\text{(rows)}\quad \begin{cases} \mathbf{b}_n^{(m)} = 3^m \mathbf{b}_n = 3^m F_{2n} \quad (= 3\mathbf{b}_n^{(m-1)}), & (3.11) \\ \mathbf{c}_n^{(m)} = 3^m \mathbf{c}_n + \mathbf{c}_0^{(m)}. & (3.12) \end{cases}$$

Binet forms:

$$\mathbf{b}_n^{(m)} = 3^m(\alpha^{2n} - \beta^{2n}) / (\alpha - \beta), \tag{3.13}$$

$$\mathbf{c}_n^{(m)} = 3^m(\alpha^{2n} + \beta^{2n}) - 2^{m+1}. \tag{3.14}$$

Generating functions:

$$\sum_{i=1}^{\infty} \mathbf{b}_i^{(m)} x^{i-1} = 3^m[1 - (3x - x^2)]^{-1}, \tag{3.15}$$

$$\sum_{i=1}^{\infty} \mathbf{c}_i^{(m)} x^{i-1} = 3^m(3 - 2x)[1 - (3x - x^2)]^{-1} - 2^{m+1}. \tag{3.16}$$

Simson formulas:

$$\mathbf{b}_{n+1}^{(m)} \mathbf{b}_{n-1}^{(m)} - (\mathbf{b}_n^{(m)})^2 = -3^{2m}, \tag{3.17}$$

$$\mathbf{c}_{n+1}^{(m)} \mathbf{c}_{n-1}^{(m)} - (\mathbf{c}_n^{(m)})^2 = 3^m\{3^m \cdot 5 - 2^{m+1} L_{2n}\}. \tag{3.18}$$

Summations:

$$\sum_{i=1}^{n} \mathbf{b}_i^{(m)} = 3^m(F_{2n+1} - 1), \tag{3.19}$$

$$\sum_{i=1}^{n} \mathbf{c}_i^{(m)} = 3^m(L_{2n+1} - 1) - 2^{m+1} n. \tag{3.20}$$

Other simple properties:

$$\mathbf{b}_n^{(m)} - \mathbf{b}_{n-1}^{(m)} = 3^m F_{2n-1}, \tag{3.21}$$

$$\mathbf{c}_n^{(m)} - \mathbf{c}_{n-1}^{(m)} = 3^m L_{2n-1}, \tag{3.22}$$

$$\mathbf{b}_n^{(m)} + \mathbf{b}_{n-1}^{(m)} = 3^m L_{2n-1} \quad \text{also}, \tag{3.23}$$

$$\mathbf{c}_n^{(m)} + \mathbf{c}_{n-1}^{(m)} = 3^m \cdot 5F_{2n-1} - 2(3^m + 2^m), \tag{3.24}$$

$$\mathbf{b}_n^{(m)} + \mathbf{c}_n^{(m)} = 2\mathcal{B}_n^{(m)}, \tag{3.25}$$

$$\mathbf{c}_n^{(m)} - \mathbf{b}_n^{(m)} = 2\mathcal{B}_{n-1}^{(m)}. \tag{3.26}$$

4. FOREGROUND

1. Augmented Sequence

Let us now recall, as in [1], the *augmented sequence* $\{\mathcal{B}_n^*(a, b, k) \equiv \mathcal{B}_n^*\}$ defined by

$$\mathcal{B}_{n+2}^*(a, b, k) = 3\mathcal{B}_{n+1}^*(a, b, k) - \mathcal{B}_n^*(a, b, k) + k. \tag{4.1}$$

Initially, assume

$$\mathcal{B}_1^*(a,b,k)=a, \quad \mathcal{B}_2^*(a,b,k)=b. \tag{4.2}$$

Hence,

$$\mathcal{B}_{n+1}^*(1,3,0)=\mathbf{b}_n, \tag{4.3}$$

and

$$\mathcal{B}_{n+1}^*(1,5,2)=\mathbf{c}_n, \tag{4.4}$$

while

$$\mathcal{B}_{n+1}^*(1,2,0)=b_n, \tag{4.5}$$

and

$$\mathcal{B}_{n+1}^*(1,4,0)=c_n. \tag{4.6}$$

2. Brahmagupta Polynomials

Very recently, Suryanarayan showed in [4] and [5] how, by means of the *Brahmagupta matrix*, to generate polynomials x_n and y_n (*Brahmagupta polynomials*) which include *inter alia* Fibonacci, Pell, and Pell-Lucas polynomials, as well as the Morgan-Voyce polynomials $B_n(x)=x_n$ and $b_n(x)=y_n$ described in [1] and [2].

Suppose we express the vital difference equations [4, eqn. (8)], [5, eqn. (9)] in a slightly varied notation as

$$x_{n+1}=Px_n-Qx_{n-1}, \quad y_{n+1}=Py_n-Qy_{n-1}. \tag{4.7}$$

Selecting $P=x+2$, $Q=1$, $x_1=2$, $x_2=P$, and $y_1=-1$, $y_2=1$ (so $y_3=x+3$) in (4.7), we readily come to the polynomials $C_n(x)=x_n$ and $c_n(x)=y_n$, which [1], [2] are adjunct to $B_n(x)$ and $b_n(x)$.

3. Further Developments

These might profitably include, for instance,

a) properties of b_{-n}, c_{-n} $(n>0)$,

b) extension of the theory to polynomials $\mathbf{b}_n(x)$, $\mathbf{c}_n(x)$ (and also $\mathcal{B}_n(x)$, $\mathcal{C}_n(x)$ [1]),

c) construction of a representation table of sufficient scope to afford numerical enhancement of the patterns contained therein,

d) uniqueness or otherwise of the representation, and

e) any additional Brahmagupta properties.

4. Associated Legendre Polynomials

The author has become aware that the Morgan-Voyce polynomials $b_n(x)$ defined in (1.1) are essentially the *associated Legendre polynomials* $\rho_n(x)$ described by Riordan [3, p. 85]. In fact, $b_{n+1}(x)=\rho_n(x)$, e.g., $b_3(x)=\rho_2(x)=1+3x+x^2$. Properties of $\rho_n(x)$ listed in [3] may then be cast in the $b_n(x)$ notation. Essential links for the equality of $\rho_n(x)$ and $b_{n+1}(x)$ are the closed forms and Chebyshev polynomials results in [3, p. 85] and [2, (2.21) and (4.14)].

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