# SIEVE FORMULAS FOR THE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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We present here two sieve-type explicit formulas for $r$-Fibonacci and $r$-Lucas numbers ($r = 2, 3, \ldots$) that connect them with families of well-defined combinatorial numbers, and discuss some particular cases.

## 1. DEFINITIONS

We consider the two main families of sequences $\{F_n^{(r)}\}$ and $\{L_n^{(r)}\}$ ($r = 2, 3, \ldots$), determined by the simplest general $r^{\text{th}}$-order linear recursion

$$Q_n^{(r)} = \sum_{k=1}^{r} Q_{n-k}^{(r)} \quad (n \geq r) \tag{1}$$

($Q_n^{(r)}$ denotes either $F_n^{(r)}$ or $L_n^{(r)}$) with initial conditions

$$F_0^{(r)} = 0,\ F_1^{(r)} = 1, \ldots,\ F_j^{(r)} = 2^{j-2} \quad (2 \leq j \leq r-1); \tag{2}$$

$$L_0^{(r)} = r,\ L_1^{(r)} = 1, \ldots,\ L_j^{(r)} = 2^j - 1 \quad (1 \leq j \leq r-1). \tag{3}$$

$F_n^{(r)}$ and $L_n^{(r)}$ are *r-Fibonacci* and *r-Lucas numbers,* respectively (cf. [2], [6], [8], [9]; also [7] with $a_i = 1$ for all $i$)—or the "fundamental" and "primordial" sequences named by Lucas. The sequences $\{F_n^{(r)}\}$ and $\{L_n^{(r)}\}$ differ from the known Tribonacci, Tetranacci, etc., sequences in having a shift $r-2$ places backwards.

The recursion (1) implies a fundamental property—the *subtraction law*

$$Q_n^{(r)} = 2Q_{n-1}^{(r)} - Q_{n-r-1}^{(r)} \quad (n \geq r+2) \tag{4}$$

for sequences of both kinds.

Our aim is to evaluate the differences $2^{n-2} - F_n^{(r)}$ and $2^n - 1 - L_n^{(r)}$ caused by this subtraction. We propose a method of exact calculation of $F_n^{(r)}$ and $L_n^{(r)}$.

As a result, explicit formulas (12) and (18) are obtained, which generalize the known formulas in the particular case $r = 2$ (Section 4).

## 2. THE $r$-FIBONACCI SEQUENCES

The evaluation of $2^{n-2} - F_n^{(r)}$ involves a family of numbers

$$\begin{aligned} d(m,1) &= 1, \\ d(m,n) &= \frac{2m+n-3}{n-1}\binom{m+n-3}{n-2}2^{n-2} \\ &= \frac{2m+n-3}{m-1}\binom{m+n-3}{m-2}2^{n-2} \quad (n \geq 2). \end{aligned} \tag{5}$$

The numbers $d(m,n)$ and $c(m,n)=d(m,n)/2^{n-2}$ for particular values of $m$ and $n$ are well known. For fixed $n$, $c(m,n)$ are the $(n-1)$-dimensional square pyramidal numbers [3] (sequences M3356, M3844, M4135, M4387 in [10]—for $n=2,3,4,5$). There is also

$$c(n-1,n)=(3n-5)C_{n-2}, \tag{6}$$

where $C_n$ is the $n^{\text{th}}$ Catalan number (M2814).

As $c(m,2)=d(m,2)=2m-1$, the array $\{c(m,n)\}$ with $m\geq 2$ may be considered as the "Pascal product" of the sequences $(1,3,5,\ldots,2m-1,\ldots)$ [or $(2,2,\ldots,2,\ldots)$, beginning from $n=1$] and $(1,1,\ldots,1,\ldots)$ with the addition law

$$c(m,n)=c(m,n-1)+c(m-1,n) \tag{7}$$

of the Pascal triangle array $\left\{\binom{m+n}{n}\right\}$.

The numbers $c(m,n)$ appear also as coefficients in the Lucas polynomials $L_n(x)$:

$$L_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor+1} c(m+1,n-2m+1)x^{n-2m}. \tag{8}$$

The numbers $d(m,n)$ enter as coefficients in the Chebyshev polynomials $T_n(x)$ [1]:

$$T_n(x)=\sum_{m=1}^{\lfloor n/2\rfloor+1}(-1)^{m-1}d(m,n-2m+3)x^{n-2m+2} \tag{9}$$

(see M2739, M3881, M4405, M4631, M4796, M4907 for $m=2,\ldots,7$).

***Proposition 1:***

$$d(m,n)=2d(m,n-1)+d(m-1,n). \tag{10}$$

***Proof:***

$$\begin{aligned}2d(m,n-1)+d(m-1,n)&=2\frac{2m+n-4}{n-2}\binom{m+n-4}{n-3}2^{n-3}+\frac{2m+n-5}{n-1}\binom{m+n-4}{n-2}2^{n-2}\\&=\left(\frac{2m+n-4}{m-1}+\frac{2m+n-5}{n-1}\right)\binom{m+n-4}{n-2}2^{n-2}\\&=\frac{2m^2+3mn+n^2-9m-6n+9}{(m-1)(n-1)}\cdot\frac{m-1}{m+n-3}\binom{m+n-3}{n-2}2^{n-2}\\&=\frac{2m+n-3}{n-1}\binom{m+n-3}{n-2}2^{n-2}=d(m,n).\quad\square\end{aligned}$$

***Theorem 1:***

$$F_n^{(r)}=\sum_{m=0}^{\lfloor\frac{n-1}{r+1}\rfloor}(-1)^m d(m+1,n-(r+1)m)\quad(n\geq 2). \tag{11}$$

***Proof:*** By induction. We see from (2) that the assertion is true for $n=2,\ldots,r+1$ because $2^{n-2}=d(1,n)$. Also, for $n=r+2$, it follows from (4) that

$$F_{r+2}^{(r)}=2F_{r+1}^{(r)}-F_1^{(r)}=2^r-1=d(1,r+2)-d(2,1).$$

Suppose that (11) holds for all $r$ previous values $n-1, \ldots, n-r-1$. Then, from (4) and (10), we obtain:

$$\begin{aligned}
F_n^{(r)} &= 2F_{n-1}^{(r)} - F_{n-r-1}^{(r)} \\
&= 2\left(2^{n-3} + \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^m d(m+1, n-(r+1)m-1)\right) \\
&\qquad - \sum_{m=0}^{\lfloor \frac{n-1}{r+1} \rfloor - 1} (-1)^m d(m+1, n-(r+1)(m+1)) \\
&= 2^{n-2} + 2\sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^m d(m+1, n-(r+1)m-1) \\
&\qquad - \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^{m-1} d(m, n-(r+1)m) \\
&= d(1,n) + \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^m d(m+1, n-(r+1)m). \quad \square
\end{aligned}$$

From (5) and (11), we obtain the resulting formula.

***Corollary 1:***

$$F_n^{(r)} = 2^{n-2} - \sum_{m=1}^{\lfloor \frac{n-1}{r+1} \rfloor} (-1)^{m-1} \frac{n-m(r-1)-1}{m} \binom{n-mr-2}{m-1} 2^{n-m(r+1)-2} \quad (n \geq 2). \tag{12}$$

***Example:***

$$\begin{aligned}
F_{22}^{(5)} &= d(1,22) - d(2,16) + d(3,10) - d(4,4) \\
&= 2^{20} - 17\binom{15}{0}2^{14} + \frac{13}{2}\binom{10}{1}2^8 - \frac{9}{3}\binom{5}{2}2^2 \\
&= 1048576 - 278528 + 16640 - 120 = 786568.
\end{aligned}$$

## 3. THE $r$-LUCAS SEQUENCES

To evaluate the difference $2^n - 1 - L_n^{(r)}$, we introduce in a similar way the numbers

$$\begin{aligned}
e(r; m, 1) &= r+1, \\
e(r; m, n) &= \frac{(r+1)m+n-1}{m} \binom{m+n-2}{n-1} 2^{n-1}.
\end{aligned} \tag{13}$$

The array $\{e(r;m,n)/2^{n-1}\}$ for the given $r$ is a Pascal product of the sequences $(r+1,\ r+1, \ldots, r+1, \ldots)$ and $(1, 1, \ldots, 1, \ldots)$ with the addition law analogous to (7). For the case $r = 2$, we find in [10] two sequences from here: M2835 ($m = 3$), M3011 ($m = 5$), explained as coefficients in the expansion of $(1-x-x^2)^{-n}$. The numbers $e(r;m,n)$ show almost no connection with the previous ones; the only common values we can notice are

$$d(2,n) = e(2; 1, n-1) = (n+1)2^{n-2}. \tag{14}$$

However, their addition properties are the same.

***Proposition 2:***

$$e(r;m,n) = 2e(r;m,n-1) + e(r;m-1,n). \tag{15}$$

***Proof:***

$$\begin{aligned}
&2e(r;m,n-1) + e(r;m-1,n)\\
&= 2\frac{(r+1)m+n-2}{m}\binom{m+n-3}{n-2}2^{n-2} + \frac{(r+1)(m-1)+n-1}{m-1}\binom{m+n-3}{n-1}2^{n-1}\\
&= \left(\frac{((r+1)m+n-2)(n-1)}{m(m-1)} + \frac{(r+1)(m-1)+n-1}{m-1}\right)\binom{m+n-3}{n-1}2^{n-1}\\
&= \frac{(r+1)m^2+(r+2)mn+n^2-(2r+3)m-3n+2}{m(m-1)}\cdot\frac{m-1}{m+n-2}\binom{m+n-2}{n-1}2^{n-1}\\
&= \frac{(r+1)m+n-1}{m}\binom{m+n-2}{n-1}2^{n-1} = e(r;m,n). \quad \square
\end{aligned}$$

For the initial value $m=1$, there obviously is

$$e(r;1,n) = 2e(r;1,n-1) + 2^{n-1}. \tag{16}$$

***Theorem 2:***

$$L_n^{(r)} = 2^n - 1 + \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor}(-1)^m e(r;m,n-(r+1)m+1) \quad (n \ge 1). \tag{17}$$

***Proof:*** By induction (as in Theorem 1). The assertion is true for the $r$ initial values (3) (for $n \ge 1$) and $L_n^{(r)} = 2^r - 1$. We also can see that

$$L_{r+1}^{(r)} = 2L_r^{(r)} - L_0^{(r)} = 2^{r+1} - 1 - (r+1) = 2^{r+1} - 1 - e(r;1,1).$$

Performing the induction step $n-1 \to n$, and using (4), (15), and (16), we obtain:

$$\begin{aligned}
L_n^{(r)} &= 2L_{n-1}^{(r)} - L_{n-r-1}^{(r)}\\
&= 2\left(2^{n-1} - 1 + \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor}(-1)^m e(r;m,n-(r+1)m)\right)\\
&\qquad - 2^{n-r-1} + 1 - \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor - 1}(-1)^m e(r;m,n-(r+1)m-r)\\
&= 2^n - 1 + \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor}(-1)^m e(r;m,n-(r+1)m)\\
&\qquad - e(r;1,n-r) - \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor - 1}(-1)^m e(r;m,n-(r+1)(m+1)+1)\\
&= 2^n - 1 + \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor}(-1)^m e(r;m,n-(r+1)m) + \sum_{m=2}^{\lfloor \frac{n}{r+1} \rfloor}(-1)^m e(r;m-1,n-(r+1)m+1)\\
&= 2^n - 1 + \sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor}(-1)^m e(r;m,n-(r+1)m+1). \quad \square
\end{aligned}$$

From (13), (17) follows

***Corollary 2:***

$$L_n^{(r)} = 2^n - 1 - n\sum_{m=1}^{\lfloor \frac{n}{r+1} \rfloor} (-1)^{m-1}\frac{1}{m}\binom{n-mr-1}{m-1}2^{n-m(r+1)} \quad (n \geq 1). \tag{18}$$

***Example:***

$$\begin{aligned} L_{20}^{(5)} &= 2^{20} - 1 - e(5;1,15) + e(5;2,9) - e(5;3,3) \\ &= 2^{20} - 1 - 20\left(\binom{14}{0}2^{14} - \frac{1}{2}\binom{9}{1}2^8 + \frac{1}{3}\binom{4}{2}2^2\right) \\ &= 1048575 - 20(16384 - 1152 + 8) = 743775. \end{aligned}$$

## 4. FORMULAS IN THE CASE $r = 2$

In the particular case $r = 2$, i.e., for the usual Fibonacci and Lucas numbers $F_n = F_n^{(2)}$, $L_n = L_n^{(2)}$ $(n \geq 3)$, the following formulas are obtained from (5), (12), (13), and (18).

***Corollary 3:***

$$\begin{aligned} F_n &= d(1,n) - d(2,n-3) + d(3,n-6) - \cdots \\ &= 2^{n-2} - \sum_{m=1}^{\lfloor (n-1)/3 \rfloor} (-1)^{m-1}\frac{n-m-1}{m}\binom{n-2m-2}{m-1}2^{n-3m-2}; \end{aligned} \tag{19}$$

$$\begin{aligned} L_n &= 2^{n-1} - 1 - e(2;1,n-2) + e(2;2,n-5) - \cdots \\ &= 2^{n-1} - n\sum_{m=1}^{\lfloor n/3 \rfloor} (-1)^{m-1}\frac{1}{m}\binom{n-2m-1}{m-1}2^{n-3m}. \end{aligned} \tag{20}$$

Formula (20) in an equivalent form was discovered by Filipponi ([4], formula (2.1)), using a simpler formula of Jaiswal [5], which (with $n$ instead of $n+3$ in the original notation) has the form

$$F_n = 1 + \sum_{m=1}^{\lfloor n/3 \rfloor} (-1)^{m-1}\binom{n-2m-1}{m-1}2^{n-3m}. \tag{21}$$

This is perhaps the first known example of Fibonacci sieve formulas.

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***Announcement***

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