ON TOTAL STOPPING TIMES UNDER 3x + 1 ITERATION

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1. INTRODUCTION

Let $\mathbf{N}$ denote the nonnegative integers, and let $\mathbf{P}$ denote the positive integers. Define T: $2\mathbf{N}+1 \to 2\mathbf{N}+1$ by $T(x) = \frac{3x+1}{2^j}$, where $2^j \mid 3x+1$ and $2^{j+1} \nmid 3x+1$. The famous $3x+1$ Conjecture asserts that, for any $x \in 2\mathbf{N}+1$, there exists $k \in \mathbf{N}$ satisfying $T^k(x) = 1$. Define the least whole number k for which $T^k(x) = 1$ as the *total stopping time* $\sigma(x)$ of x, and call the sequence of iterates $(x, T(x), T^2(x), \ldots)$ the *trajectory* of x. Note that $\sigma(x) = \infty$ if the trajectory of x diverges, and that $\sigma(1) = 0$. Furthermore, if $k \in \mathbf{P}$ is fixed, and x is the smallest positive odd integer satisfying $T^k(x) = 1$, we say that x is *minimal of level k*. In this paper, we employ a specific partition of the positive odd integers to show that if x is minimal of level $k \geq 3$, then $\sigma(x) = \sigma(2x+1)$. In addition, a set of positive integers satisfying $\sigma(x) = \sigma(2x+1)$ is characterized. Using a related partition, we then show that the arithmetic progression (1 *mod* 16) is a "sufficient set," in other words, to prove the $3x+1$ Conjecture, it suffices to prove it for all $x \equiv 1$ *mod* 16. In [4], Korec and Znam proved that the arithmetic progressions (a *mod* p^n), where 2 is a primitive root (*mod* p^2) and $(a, p) = 1$, are sufficient sets; however, this result does not apply when p is a power of 2.

A thorough summary of some known results on the $3x+1$ Conjecture is given in Lagarias [5] and Wirsching [6]. It is important to observe that our formulation of the function $T(x)$ differs from that in [3], in which $T: \mathbf{P} \to \mathbf{P}$ is given by $T(x) = \frac{x}{2}$ if x is even and $T(x) = \frac{3x+1}{2}$ is x is odd. As a consequence, our total stopping times are different. For example, $\sigma(27) = 41$ under our formulation, whereas $\sigma(27) = 70$ in [3].

It is the author's hope that the results of this paper, or perhaps the techniques used in proving the results, will be useful in computing $\pi_a(x)$, which counts the number of positive integers $y \leq x$ such that $T^k(y) = a$ for some nonnegative integer k. The strongest known results along this line are given in Applegate and Lagarias [1].

2. TOTAL STOPPING TIMES OF MINIMAL NUMBERS

We begin by constructing a partition of the positive odd integers. For $a, b \in \mathbf{P}$, denote the arithmetic progression $(\mathrm{am}+\mathrm{b})_{\mathrm{m}=0}^{\infty}$ by $(\mathrm{am}+\mathrm{b})$. Next, define subsets of $2\mathbf{N}+1$ as follows:

$$S_1 = \bigcup_{n \in \mathbf{P}} (2^{2n+1}m + 2^{2n-1} - 1),$$

$$S_2 = \bigcup_{n \in \mathbf{P}} (2^{2n+2}m + 2^{2n+1} + 2^{2n} - 1),$$

$$S_3 = \bigcup_{n \in \mathbf{P}} (2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1),$$

$$S_4 = \bigcup_{n \in \mathbf{P}} (2^{2n+2}m + 2^{2n} - 1).$$

It is easy to verify that $[S_1, S_2, S_3, S_4]$ is a partition of $2\mathbf{N}+1$. We will also need the following two preliminary lemmas, both of which follow directly from the definition of $T(x)$.

Lemma 1: Let $x \in 2\mathbf{N}+1$, and let $k \in \mathbf{N}$ satisfy $k \le \sigma(x)$. Then $\sigma(T^k(x)) = \sigma(x) - k$.

Lemma 2: Let $x \in 2\mathbf{N}+1$ with $x \neq 1$. Then $\sigma(x) = \sigma(4x+1)$.

The following two lemmas give total stopping time properties of certain subsets of the positive integers obtained from our partition. For notational convenience in the upcoming proofs and throughout this paper, we write $2^j \| n$ (2^j *exactly divides* n) if $2^j \mid n$ but $2^{j+1} \nmid n$.

Lemma 3: If $x \in S_1 \cup S_2 - (1)$, then $\sigma(x) = \sigma(2x+1)$.

Proof: First, consider the case in which $x \in S_1$ with $x \neq 1$. By the definition of S_1, x is of the form $2^{2n+1}m + 2^{2n-1} - 1$. Application of the function T yields:

$$T^{2n-1}(x) = \frac{3^{2n-1} \cdot 4m + 3^{2n-1} - 1}{2^j},$$

where $2^j \| 3^{2n-1} \cdot 4m + 3^{2n-1} - 1$. Note that $3^{2n-1} - 1 \equiv 2 \ mod\ 4$, therefore $j = 1$. Furthermore, $T^{2n-1}(2x+1) = 3^{2n-1} \cdot 8m + 3^{2n-1} \cdot 2 - 1$. Thus, $4 \cdot T^{2n-1}(x) + 1 = T^{2n-1}(2x+1)$. Applying Lemma 2, we obtain $\sigma(T^{2n-1}(x)) = \sigma(T^{2n-1}(2x+1))$. Hence, by Lemma 1, it follows that $\sigma(x) = \sigma(2x+1)$.

Next, consider the case $x \in S_2$. By definition of S_2, x is of the form $2^{2n+2}m + 2^{2n+1} + 2^{2n} - 1$. Application of the function T yields:

$$T^{2n}(x) = \frac{3^{2n} \cdot 4m + 3^{2n} \cdot 2 + 3^{2n} - 1}{2^j},$$

where $2^j \| 3^{2n} \cdot 4m + 3^{2n} \cdot 2 + 3^{2n} - 1$. Since $3^{2n} - 1 \equiv 0 \ mod\ 4$ and $3^{2n} \cdot 2 \equiv 2 \ mod\ 4$, we see that $j = 1$. Furthermore, $T^{2n}(2x+1) = 3^{2n} \cdot 8m + 3^{2n} \cdot 4 + 3^{2n} \cdot 2 - 1$. Hence, $4 \cdot T^{2n}(x) + 1 = T^{2n}(2x+1)$. Applying Lemma 2 yields $\sigma(T^{2n}(x)) = \sigma(T^{2n}(2x+1))$, so, using Lemma 1, we conclude that $\sigma(x) = \sigma(2x+1)$. □

Lemma 4: If $x \in S_3 \cup S_4 - (3)$, then there exists $y < x$ satisfying $\sigma(y) = \sigma(x)$.

Proof: First, consider the case in which $x \in S_3$. By definition of S_3, we have $x = 2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1$. If $n = 1$, $x = 8m + 5$, so choosing $y = 2m + 1$ and applying Lemma 2 gives the result. If $n > 1$, we can choose $y \in 2\mathbf{N}+1$ satisfying $2y + 1 = x$. Note that $y \in S_2$, so using a computation similar to that in the proof of Lemma 3, we see that $4 \cdot T^{2n-2}(y) + 1 = T^{2n-2}(x)$. Applying Lemmas 2 and 1, we obtain $\sigma(y) = \sigma(x)$. Now consider the case in which $x \in S_4$ with $x \neq 3$. By definition of S_4, we have $x = 2^{2n+2}m + 2^{2n} - 1$. Again, choose y so that $2y + 1 = x$. Clearly, $y \in S_1$, so again by the proof of Lemma 3, it follows that $4 \cdot T^{2n-1}(y) + 1 = T^{2n-1}(x)$. Noting that $y \neq 1$ and applying Lemmas 1 and 2, we obtain $\sigma(y) = \sigma(x)$. □

The following result pertaining to total stopping times of minimal numbers can now be proved.

Theorem 1: If x is minimal of level $k \ge 3$, then $\sigma(x) = \sigma(2x+1)$.

Proof: Let $x \in 2\mathbf{N}+1$ be minimal of level $k \geq 3$. Note that $x \neq 1$ and $x \neq 3$. Using the definition of minimality and Lemma 4, we see that $x \notin S_3 \cup S_4$. Therefore $x \in S_1 \cup S_2$, so Lemma 3 implies that $\sigma(x) = \sigma(2x+1)$. □

Remark: The arguments in Lemmas 3 and 4 actually show that the appropriate trajectories coalesce after a certain number of steps, irrespective of whether or not they converge to 1. This is in part due to the fact that if $f(x) = 4x+1$ and x is odd, then $T(f(x)) = T(x)$. Note also that if $g(x) = 2x+1$, the relation $T(g(x)) = g(T(x))$ holds true for x odd. Furthermore, it can be demonstrated by straightforward computation that if $g_{a,b}(x) = ax+b$ with $a-b=1$ and x is of the form $2^n m + 2^{n-2} - 1$ or $2^n m + 2^{n-1} + 2^{n-2} - 1$ with $n \geq 3$, then $g_{a,b}(T^k(x)) = T^k(g_{a,b}(x))$ for $k \leq n-3$. A study of the interaction of various linear functions $g_{a,b}(x)$ with $T(x)$ under composition deserves further exploration.

3. A SUFFICIENT CONDITION FOR TRUTH OF THE $3x+1$ CONJECTURE

By use of a similar technique, it can now be demonstrated that to prove the $3x+1$ Conjecture, it suffices to prove it for all positive $x \equiv 1 \; mod \; 16$. This improves a result given in Cadogan [2].

Lemma 5: Suppose that for all positive $x \equiv 1 \; mod \; 8$ there exists $k \in \mathbf{N}$ such that $T^k(x) = 1$. Then, for all $x \in 2\mathbf{N}+1$. we can find $k \in \mathbf{N}$ such that $T^k(x) = 1$.

Proof: For $i = 1, 2, 3, 4$, define $T_i = S_i \cap (8m+7)$, where $[S_1, S_2, S_3, S_4]$ is the partition of $2\mathbf{N}+1$ used in Lemmas 3 and 4. We repartition the positive odd integers as follows:

$$2\mathbf{N}+1 = (8m+1) \cup (16m+3) \cup (16m+11) \cup (8m+5) \cup T_1 \cup T_2 \cup T_3 \cup T_4.$$

Now let $x \in 2\mathbf{N}+1$ be given. We can assume that $x \neq 1$ and $x \neq 3$, as the theorem follows trivially for these values of x. We examine the following cases:

Case 1. If $x \in (8m+1)$, by the hypothesis of Lemma 5, there exists $k \in \mathbf{P}$ such that $T^k(x) = 1$.

Case 2. Let $x \in (16m+3)$. Then $x = 2y+1$ for $y \in (8m+1)$. A simple computation shows that $T^2(x) = T^2(y)$. By the hypothesis of Lemma 5, there exists $k \in \mathbf{P}$ such that $T^k(y) = 1$, hence $T^k(x) = 1$.

Case 3. Let $x \in (16m+11)$. Then $T(x) \in (8m+1)$, so the hypothesis of Lemma 5 guarantees that there exists $k \in \mathbf{P}$ satisfying $T^k(T(x)) = 1$. Thus, $T^{k+1}(x) = 1$.

Case 4. Let $x \in T_1 \cup T_2$. If $x \in T_1$, we can write $x = 2^{2n+1}m + 2^{2n-1} - 1$, where $n \geq 2$. Then $T^{2n-2}(x) = 3^{2n-2} \cdot 8m + 3^{2n-2} \cdot 2 - 1$, and since $3^{2n-2} \equiv 1 \; mod \; 8$, we see that $T^{2n-2}(x) \in (8m+1)$. If $x \in T_2$, we can write $x = 2^{2n+2}m + 2^{2n+1} + 2^{2n} - 1$, where $n \geq 2$. Then $T^{2n-1}(x) = 3^{2n-1} \cdot 8m + 3^{2n-1} \cdot 4 + 3^{2n-1} \cdot 2 - 1$, which simplifies to $T^{2n-1}(x) = 3^{2n-1} \cdot 8m + 2(3^{2n} - 1) + 1$, and since $3^{2n} - 1 \equiv 0 \; mod$ 4, we obtain $T^{2n-1}(x) \in (8m+1)$. Invoking our hypothesis yields $T^k(x) = 1$ for some k.

Case 5. Let $x \in T_3 \cup T_4$. If $x \in T_3$, then x is of the form $2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1$, where $n \geq 2$. Choose y satisfying $2y+1 = x$. By a computation similar to that used in the proof of Lemma 4, we see that $4 \cdot T^{2n-2}(y) + 1 = T^{2n-2}(x)$, hence $T^{2n-1}(y) = T^{2n-1}(x)$. If $n = 2$, $y \in (16m+11)$, and if $n > 2$, $y \in T_2$, so by the proofs of Case 3 and Case 4, respectively, there exists k satisfying

$T^k(y)=1$, hence $T^k(x)=1$. If $x \in T_4$, then x is of the form $2^{2n+2}m+2^{2n}-1$, where $n \geq 2$. Let y satisfy $2y+1=x$. Again, $4 \cdot T^{2n-1}(y)+1=T^{2n-1}(x)$, so $T^{2n}(y)=T^{2n}(x)$. But $y \in T_1$, so by Case 4, there exists k satisfying $T^k(y)=1$, hence $T^k(x)=1$.

Case 6. Finally, let $x \in (8m+5)$. Define $f(w)=4w+1$. Choose the smallest positive y satisfying $f^n(y)=x$ for $n \in \mathbf{P}$. Note that $y \notin (8m+5)$, since $f(2m+1)=8m+5$. If $y \neq 1$ and $y \neq 3$, we can invoke the previous cases to obtain k satisfying $T^k(y)=1$. Since $T(f^n(y))=T(y)$, we obtain $T(y)=T(x)$, and therefore $T^k(y)=T^k(x)=1$. If $y=3$, then $T(f^n(y))=T(y)=T(3)=5$, hence $T^2(f^n(y))=1$, so $T^2(x)=1$. If $y=1$, we have $f^n(y)=1+4+\cdots+4^n=(4^{n+1}-1)/3$, hence $T(f^n(y))=1$, so $T(x)=1$. Thus, in all cases, we have displayed $k \in \mathbf{N}$ for which $T^k(x)=1$. □

According to Lemma 5, the arithmetic progression $(8m+1)$ constitutes a sufficient set. The next theorem improves the sufficient set.

Theorem 2: Suppose that for all positive $x \equiv 1 \; mod \; 16$, there exists $k \in \mathbf{N}$ such that $T^k(x)=1$. Then, for all $x \in 2\mathbf{N}+1$, we can find $k \in \mathbf{N}$ such that $T^k(x)=1$.

Proof: Let $x=8m+1$ be given. A straightforward computation yields

$$T^2(64x+49)=\frac{9x+7}{2^j}=\frac{72m+16}{2^j}=\frac{9m+2}{2^{j-3}},$$

where $2^j \,\|\, 9x+7$, and hence $2^{j-3} \,\|\, 9m+2$. Also,

$$T^2(x)=T^2(8m+1)=\frac{9m+2}{2^k},$$

where $2^k \,\|\, 9m+2$. By unique factorization, $k=j-3$, and hence $T^2(x)=T^2(64x+49)$. Since $64x+49$ is in the arithmetic progression $(16m+1)$, we can invoke the hypothesis of Theorem 2; therefore, there exists k satisfying $T^k(T^2(x))=1$. Thus, $T^{k+2}(x)=1$, and since x was chosen arbitrarily from $(8m+1)$, we can apply Lemma 5 to obtain the result. □

Further strengthening of the result given in Theorem 2 certainly seems possible. An interesting question concerns which progressions of the form $(2^n m+1)$ constitute "sufficient sets" whose convergence to 1 guarantees the truth of the $3x+1$ Conjecture. Perhaps it can be proved that convergence of the set of numbers of the form $\{2^n+1 : n=1,2,3,\ldots\}$ is sufficient.

4. OTHER NUMBERS WITH EQUAL TOTAL STOPPING TIMES

We now characterize an additional set of positive odd integers satisfying $\sigma(x)=\sigma(2x+1)$. Let $L_k=\{x \in 2\mathbf{N}+1 \mid \sigma(x)=k\}$, and define $G_x=\{f^n(x) \mid n \in \mathbf{N}\} \cup \{f^n(2x+1) \mid n \in \mathbf{N}\}$, where $f(w)=4w+1$. For convenience, we set $G_{x_{-1}}=\emptyset$. We inductively define the j^{th} *exceptional number of level* k to be the smallest positive integer x_j satisfying $x_j \in L_k - \bigcup_{i=0}^{j} G_{x_{i-1}}$.

Note that for $j=0$, x_j is simply the minimal number of level k. Also observe that Lemma 2 and Theorem 1 tell us that all numbers in G_{x_0} are of level k, hence x_1 is the smallest positive integer of level k not accounted for by G_{x_0}, x_2 is the smallest positive integer of level k not accounted for by $G_{x_0} \cup G_{x_1}$, and so forth. It turns out that the exceptional numbers share the same total stopping time property as the minimal numbers.

Theorem 3: Let x_j denote the j^{th} exceptional number of level k with $k \geq 2$ and $x_j > 3$. Then $\sigma(x_j) = \sigma(2x_j + 1)$.

To prove Theorem 3, we need the following two preliminary lemmas.

Lemma 6: Let x_j denote the j^{th} exceptional number of level k with $k \geq 2$ and $x_j > 3$. Then $x_j \notin (16m+3) \cup (8m+5)$

Proof: Since x_0 is minimal of level k with $k \geq 2$ and $x_j > 3$, we have $x_0 \notin (16m+3) \cup (8m+5)$, hence the Lemma holds for $j = 0$. Let $j \geq 1$. We prove that $x_j \notin (16m+3)$ by contradiction. If $x_j \in (16m+3)$, pick y satisfying $2y+1 = x_j$. Clearly $\sigma(y) = \sigma(x_j)$, hence $y \in L_k$. Since $y < x_j$ and x_j is the smallest number in $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}}$, we see that $y \in G_{x_i}$ for some $i \leq j-1$. Hence $y = f^p(x_i)$ or $y = f^p(2x_i+1)$ for some $p \in \mathrm{N}$. Since $p \geq 1$ yields $y \in (8m+5)$, which is impossible, we have $p = 0$. Hence $y = x_i$ or $y = 2x_i + 1$. But $y = x_i$ yields $2x_i + 1 = x_j$, so $x_j \in G_{x_i}$ with $i \leq j-1$, contradicting the definition of x_j. Hence $y = 2x_i + 1$. But $y \in (8m+1)$ forces x_i to be even, again a contradiction. If $x_j = 8m+5$, then select $y = 2m+1$. Since $\sigma(y) = \sigma(x_j)$ and $y < x_j$, we see that $y \in G_{x_i}$ for some $i \leq j-1$. But $x_j = f(y)$, hence $x_j \in G_{x_i}$, contradicting the definition of x_j. Hence $x_j \notin (8m+5)$ □

Lemma 7: Let S_3 and S_4 be subsets of $2\mathrm{N}+1$ as defined in Section 2. Let x_j be the j^{th} exceptional number of level k with $k \geq 2$ and $x_j > 3$. Then $x_j \notin S_3 \cup S_4$.

Proof: Suppose $x_j \in S_3 \cup S_4$. Then x_j is of the form $2^{2n+1}m + 2^{2n} + 2^{2n-1} - 1$ or $2^{2n+2}m + 2^{2n} - 1$. Furthermore, by Lemma 6, we have $n \geq 2$. Choose y satisfying $2y+1 = x_j$. As in the proof of Lemma 4, we have $\sigma(y) = \sigma(x_j)$, therefore, by definition of x_j, we must have $y \in G_{x_i}$ for some $i \leq j-1$. Therefore, $y = f^p(x_i)$ or $y = f^p(2x_i+1)$ for some $p \in \mathrm{N}$. If $p \geq 1$, we have $y \in (8m+5)$, hence $x_j \in (16m+11)$, which contradicts the fact that $S_3 \cup S_4$ and $(16m+11)$ are disjoint. Thus $p = 0$, so either $y = x_i$ or $y = 2x_i + 1$. But $y = x_i$ yields $2x_i + 1 = x_j$, hence $x_j \in G_{x_i}$ for $i \leq j-1$, contradicting the definition of x_j. Thus, we have $y = 2x_i + 1$, so $4x_i + 3 = x_j$.

A simple computation shows that x_i must be in $S_3 \cup S_4$. We therefore have proven that $x_j \in S_3 \cup S_4$ implies there exists $x_i \in S_3 \cup S_4$ with $x_i < x_j$. Applying a simple induction and using the definition of S_3 and S_4 yields $x_p \in (8m+5) \cup (16m+3)$ for some p. But this contradicts Lemma 6, hence $x_j \in S_3 \cup S_4$ is impossible. □

Proof of Theorem 3: Consider the partition of $2\mathrm{N}+1$ as defined in the proof of Lemma 5. By Lemmas 6 and 7, we see that $x_j \notin (16m+3) \cup (8m+5) \cup T_3 \cup T_4$. Hence $x_j \in (8m+1) \cup (16m+11) \cup T_1 \cup T_2$. Applying Lemma 3, we obtain $\sigma(x_j) = \sigma(2x_j+1)$. □

Our final theorem enables us to conclude that there exists an exceptional number x_j of level k for all $k \geq 2$ and for all $j \geq 0$.

Theorem 4: For all $j \geq 0$ and $k \geq 2$, $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}} \neq \emptyset$.

Proof: We proceed by induction on j. Since $L_k \neq \emptyset$ is well known [3], the result holds true for $j = 0$. Now assume $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}} \neq \emptyset$ for all $j < n$. We wish to show that $L_k - \bigcup_{i=0}^{n} G_{x_{i-1}} \neq \emptyset$. For all $j < n$, let x_j be the smallest integer in $L_k - \bigcup_{i=0}^{j} G_{x_{i-1}}$. Note that the sequence $\{x_j\}$ is strictly increasing, and that $x_j \notin G_{x_i}$ for $i \leq j-1$.

Consider the number $w = 64x_{n-1}+49$. We first prove that $w \notin G_{x_i}$ for all $i \le n-1$ by contradiction. If $w \in G_{x_i}$ for some $i \le n-1$, then $w = f^p(x_i)$ or $w = f^p(2x_i+1)$ for some $p \in \mathbf{N}$. Since $w \in (8m+1)$, we must have $p = 0$. Therefore, $w = x_i$ or $w = 2x_i+1$, and since the latter contradicts oddness of x_i, we have $w = x_i$. But this implies that $x_{n-1} < x_i$, contradicting the fact that $\{x_j\}$ is strictly increasing. Hence $w \notin G_{x_i}$ for all $i \le n-1$. Furthermore, as seen in the proof of Theorem 3, we have $\sigma(w) = \sigma(x_{n-1}) = k$, hence w is in $L_k - \bigcup_{i=0}^{n} G_{x_{i-1}}$, so $L_k - \bigcup_{i=0}^{n} G_{x_{i-1}} \neq \emptyset$. □

Remark: An interesting question concerns whether *all* numbers x satisfying $\alpha(x) = \sigma(2x+1)$ can be identified. The general question of finding all numbers x satisfying $\sigma(x) = \sigma(ax+b)$ for arbitrary whole numbers a and b looks difficult. Development of functions such as $f(w) = 64w+49$ which satisfy the condition $\sigma(x) = \sigma(f(x))$ appears to be a promising approach.

REFERENCES

1. D. Applegate & J. C. Lagarias. "Density Bounds for the $3x+1$ Problem. II. Krasikov Inequalities." *Math. Comp.* **64** (1995):411-27.
2. C. C. Cadogan. "A Note on the $3x+1$ Problem." *Caribbean J. Math.* **3** (1984):67-72.
3. R. E. Crandall. "On the '$3x+1$' Problem." *Math. Comp.* **32** (1978):1281-92.
4. I. Korec & S. Znam. "A Note on the $3x+1$ Problem." *Amer. Math. Monthly* **94.8** (1987): 771-72.
5. J. C. Lagarias. "The $3x+1$ Problem and Its Generalizations." *Amer. Math. Monthly* **92** (1985):3-21.
6. G. J. Wirsching. *The Dynamical System Generated by the $3n+1$ Function*. Lecture Notes in Math. **1681**. New York: Springer-Verlag, 1998.

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