ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ, 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net on the Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;
\]
\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]

Also, $\alpha = (1 + \sqrt{5}) / 2$, $\beta = (1 - \sqrt{5}) / 2$, $F_n = (\alpha^n - \beta^n) / \sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-889 Proposed by Mario DeNobili, Vaduz, Lichtenstein
Find 17 consecutive Fibonacci numbers whose average is a Lucas number.

B-890 Proposed by Stanley Rabinowitz, Westford, MA
If $F_a F_b F_{a-b} + F_{-a} F_{-b} F_{a+b} + F_{-c} F_{-d} F_{c+d} = 0$, show that either $a = b$, $b = c$, or $c = a$.

B-891 Proposed by Aloysius Dorp, Brooklyn, NY
Let $(P_n)$ be the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Find integers $a$, $b$, and $m$ such that $P_n = P_{an+b} \pmod{m}$ for all integers $n$.

B-892 Proposed by Stanley Rabinowitz, Westford, MA
Show that, modulo 47, $F_n^2 - 1$ is a perfect square if $n$ is not divisible by 16.

B-893 Proposed by Aloysius Dorp, Brooklyn, NY
Find integers $a$, $b$, $c$, and $d$ so that

\[
F_x F_{x+1} F_{x+2} + aF_{x+1} F_{x+2} F_{x+3} + bF_{x+2} F_{x+3} F_{x+4} + cF_{x+3} F_{x+4} F_{x+5} + dF_{x+4} F_{x+5} F_{x+6} = 0
\]

is true for all $x$, $y$, and $z$.

B-894 Proposed by the editor
Solve for $x$:

\[
F_{110}^x + 442 F_{115}^x + 13 F_{119}^x = 221 F_{114}^x + 255 F_{117}^x.
\]
SOLUTIONS

Absolute Sum

B-871 Proposed by Paul S. Bruckman, Berkeley, CA
(Vol. 37, no. 1, February 1999)

Prove that

\[ \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^3 = n^2 \binom{2n}{n}. \]

Solution by Indulis Strazdins, Riga Technical University, Latvia

The sum is equal to

\[ S(n) = 2 \sum_{k=0}^{n-1} (n-k)^3 \binom{2n}{k} = 2n^3s_0 - 6n^2s_1 + 6ns_2 - 2s_3, \]

where the expressions

\[ s_m = \sum_{k=0}^{n-1} k^m \binom{2n}{k} \quad (m = 0, 1, 2, 3) \]

can be derived from the known formulas

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n, \]
\[ \sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}, \]
\[ \sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1) \cdot 2^{n-2}, \]
\[ \sum_{k=0}^{n} k^3 \binom{n}{k} = n^2(n+3) \cdot 2^{n-3}. \]

The results are

\[ s_0 = 2^{2n-1} - \frac{1}{2} \binom{2n}{n}, \]
\[ s_1 = n \left( 2^{2n-1} - \binom{2n}{n} \right), \]
\[ s_2 = n \left( (2n+1)2^{2n-2} - \frac{3}{2} n \binom{2n}{n} \right), \]
\[ s_3 = n^2 \left( (2n+3)2^{2n-2} - \frac{1}{2} (4n+1) \binom{2n}{n} \right). \]

Thus,

\[ S(n) = (4n^3 - 12n^2 + 6n^2(2n+1) - 2n^2(2n+3))2^{2n-2} - (n^3 - 6n^2 + 9n^3 - n^2(4n+1)) \binom{2n}{n} = n^2 \binom{2n}{n}. \]
Bruckman noted that
\[ \sum_{k=0}^{2n} \binom{2n}{k} |n-k| = n \binom{2n}{n} \]
and conjectures that
\[ \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^{2r-1} = P_r(n) \binom{2n}{n} \]
for some monic polynomial \( P_r(n) \) of degree \( r \).

Solutions also received by H.-J. Seiffert and the proposer.

Rational Recurrence

B-872 Proposed by Murray S. Klamkin, University of Alberta, Canada
(Vol. 37, no. 2, May 1999)

Let \( r_n = F_{n+1} / F_n \) for \( n > 0 \). Find a recurrence for \( r_n = r_n^2 \).

Solution 1 by Maitland A. Rose, University of South Carolina, Sumter, SC

\[ t_n = \frac{F_{n+1}^2}{F_n^2} = \frac{F_n^2 + 2F_n F_{n-1} + F_{n-1}^2}{F_n^2} = 1 + \frac{2F_{n-1} + F_{n-2}}{F_n} = 1 + \frac{2}{\sqrt{F_{n-1}}} + \frac{1}{t_{n-1}}. \]

Solution 2 by Kathleen E. Lewis, SUNY, Oswego, NY

The identity \( F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \) is straightforward to prove. Dividing by \( F_n^2 \) gives

\[ t_n = 2 + \frac{2}{t_{n-1}} - \frac{1}{t_{n-1} t_{n-2}}. \]

Klamkin, Morrison, and Seiffert all found the corresponding recurrence for an arbitrary second-order linear recurrence \( w_{n+2} = Pw_{n+1} - Qw_n \). If \( t_n = (w_{n+1} / w_n)^2 \), then

\[ t_n = (P^2 - Q) + \frac{(P^2 - Q)O}{t_{n-1}} + \frac{O^3}{t_{n-1} t_{n-2}}. \]

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, John F. Morrison, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

A Property of 3

B-873 Proposed by Herta Freitag, Roanoke, VA
(Vol. 37, no. 2, May 1999)

Prove that 3 is the only positive integer that is both a prime number and of the form \( 3n + (-1)^n L_n \).

Solution by L. A. G. Dresel, Reading, England

Put \( T_n = L_{3n} + (-1)^n L_n \). Since the Binet forms for \( L_{3n} \) and \( L_n \) give the identity \( L_{3n} = L_n^2 - 3(-1)^n L_n \), we have \( T_n = L_n(L_n^2 - 2(-1)^n) = L_n L_{2n} \). Now \( L_n = 1 \) only if \( n = 1 \), so that \( T_1 = 3 \). But when \( n \neq 1 \), \( T_n \) is the product of two integers, each greater than 1. Hence, 3 is the only prime of the form \( T_n \).
**Another Property of 3**

**B-874 Proposed by David M. Bloom, Brooklyn College, NY**

(Vol 37, no. 2, May 1999)

Prove that 3 is the only positive integer that is both a Fibonacci number and a Mersenne number. [A Mersenne number is a number of the form \(2^n - 1\).]

**Solution by the proposer**

If \(F_n = 2^a - 1\) with \(a \geq 2\), then \(F_n + 1 = 2^a\). But the general identity \(F^{a+b} + (-1)^bF^{a-b} = F^aL_b\) shows that

\[
\begin{align*}
  n = 4k & \quad \text{implies} \quad F_n + 1 = F_{2k-1}L_{2k+1}, \\
  n = 4k + 1 & \quad \text{implies} \quad F_n + 1 = F_{2k+1}L_{2k}, \\
  n = 4k + 2 & \quad \text{implies} \quad F_n + 1 = F_{2k+2}L_{2k}, \\
  n = 4k + 3 & \quad \text{implies} \quad F_n + 1 = F_{2k+1}L_{2k+2}.
\end{align*}
\]

Thus, if \(F_n + 1 = 2^a\), the \(L\)-factor on the right must be a power of 2. But it must also be less than or equal to 4 since no Lucas number is divisible by 8. Thus, in all cases, \(L_{2k} \leq 4\) and \(k \geq 1\) since \(F_n \geq 3\). Hence, \(k = 1\) and the result follows.

**Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, and H.-J. Seiffert.**

**A Third Property of 3**

**B-875 Proposed by Richard André-Jeannin, Cosnes et Romain, France**

(Vol 37, no. 2, May 1999)

Prove that 3 is the only positive integer that is both a triangular number and a Fermat number. [A triangular number is a number of the form \(n(n+1)/2\). A Fermat number is a number of the form \(2^n + 1\).]

**Solution by H.-J. Seiffert, Berlin**

Let \(n\) be a positive integer and \(a\) a nonnegative integer such that \((n+1)/2 = 2^a + 1\). Multiplying by 2 and then subtracting 2 on both sides yields \((n-1)(n+2) = 2^{a+1}\). Hence, \(n \geq 2\), and \(n-1\) and \(n+2\) both must be powers of 2. Since \(n-1\) and \(n+2\) are of opposite parity, we then must have \(n-1 = 2^0\) or \(n = 2\). This gives \((n+1)/2 = 3 = 2^1 + 1\).

**Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Jaroslav Seibert, and the proposer.**

**Trigonometric Sum**

**B-876 Proposed by N. Gauthier, Royal Military College of Canada**

(Vol 37, no. 2, May 1999)

Evaluate

\[
\sum_{k=1}^{n} \sin \left( \frac{nF_{k-1}}{F_kF_{k+1}} \right) \sin \left( \frac{nF_{k+2}}{F_kF_{k+1}} \right).
\]
**Solution by Jaroslav Seibert, University of Education, Czech Republic**

For all real numbers $x$ and $y$, we have

$$\sin \frac{x+y}{2} \sin \frac{x-y}{2} = - \frac{1}{2} (\cos x - \cos y).$$

Therefore,

$$\sum_{k=1}^{n} \sin \left( \frac{nF_{k-1}}{F_k F_{k+1}} \right) \sin \left( \frac{nF_{k+2}}{F_k F_{k+1}} \right) = \sum_{k=1}^{n} \sin \left( \frac{F_{k+1} - F_k}{F_k F_{k+1}} \right) \sin \left( \frac{F_{k+1} + F_k}{F_k F_{k+1}} \right)$$

$$= - \frac{1}{2} \sum_{k=1}^{n} \left( \cos 2\pi \frac{F_{k+1}}{F_k F_{k+1}} - \cos 2\pi \frac{F_k}{F_k F_{k+1}} \right)$$

$$= - \frac{1}{2} \sum_{k=1}^{n} \left( \cos 2\pi \frac{1}{F_k} - \cos 2\pi \frac{1}{F_{k+1}} \right)$$

$$= - \frac{1}{2} \left( \cos 2\pi \frac{1}{F_{n+1}} - 1 \right) = - \frac{\sin^2 \pi}{F_{n+1}}.$$

**Solutions also received by Paul S. Bruckman, Charles K. Cook, Mario DeNobili, Leonard A. G. Dresel, John F. Morrison, Maitland A. Rose, H.-J. Seiffert, and the proposer.**

**Determining the Determinant**

**B-877** Proposed by Indulis Strazdins, Riga Technical University, Latvia

(Vol. 37, no. 2, May 1999)

Evaluate

$$\begin{vmatrix}
F_n F_{n+1} & F_{n+2} F_{n+2} & F_n F_{n+3} & F_{n+2} F_{n+4} \\
F_{n+4} F_{n+5} & F_{n+6} F_{n+6} & F_{n+4} F_{n+7} & F_{n+6} F_{n+8} \\
F_{n+8} F_{n+9} & F_{n+10} F_{n+10} & F_{n+8} F_{n+11} & F_{n+10} F_{n+12} \\
F_{n+12} F_{n+13} & F_{n+14} F_{n+14} & F_{n+12} F_{n+15} & F_{n+14} F_{n+16}
\end{vmatrix}$$

**Solution by the proposer**

Let $P_n = F_n F_{n+1}$. It is straightforward to prove the identity

$$P_{n+3} = 2P_{n+2} + 2P_{n+1} - P_n.$$

Hence, the 4th column is a linear combination of the first three ones, and therefore the determinant is 0.

**Most of the solvers pointed out analogous results for larger determinants. If the determinant contains the product of $k$ Fibonacci numbers, $F_{n+1} F_{n+2} \ldots F_{n+k-1}$, then the determinant is 0 when the order of the determinant is at least $k+2$.**

**Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Jaroslav Seibert, H.-J. Seiffert, and the proposer.**

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**Harmonic Inequality**

**B-878**  
*Proposed by L. A. G. Dresel, Reading, England*  
*(Vol. 37, no. 3, August 1999)*

Show that, for positive integers \( n \), the harmonic mean of \( F_n \) and \( L_n \) can be expressed as the ratio of two Fibonacci numbers, and that it is equal to \( L_{n-1} + R_n \), where \( |R_n| \leq 1 \). Find a simple formula for \( R_n \).

**Note:** If \( h \) is the harmonic mean of \( x \) and \( y \), then \( 2/h = 1/x + 1/y \).

**Solution by Harris Kwong, SUNY College at Fredonia, NY**

The harmonic mean of \( F_n \) and \( L_n \) is given by

\[
\frac{2F_nL_n}{F_n + L_n} = \frac{2F_{2n}}{F_{2n+1} + F_{n+1}} = \frac{F_{2n}}{F_{n+1}} = L_{n-1} + (-1)^n, 
\]

in which \( F_{2n} = F_{n+1}L_{n-1} + (-1)^n \) follows from Binet's formulas.

**Solutions also received by** Paul S. Bruckman, Charles K. Cook, Don Redmond, H.-J. Seiffert, James A. Sellers, Indulis Strazdins, and the proposer.

**Addenda.** We wish to belatedly acknowledge solutions from the following solvers:

Brian Beasley solved B-854, 855, 857, 860, 862, 863, and 864.
L. A. G. Dresel solved B-866, 867, 868, 869, and 870.

\[\bigstar\bigstar\bigstar\]