SUMS OF CERTAIN PRODUCTS OF FIBONACCI AND LUCAS NUMBERS–PART II

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1. INTRODUCTION

The identities

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1} \tag{1.1}$$

and

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 = L_n L_{n+1} - L_0 L_1 \tag{1.2}$$

are well known. The right side of (1.2) suggests the notation $[L_j L_{j+1}]_0^n$, which we use throughout this paper in order to conserve space. Each time we use this notation, we take j to be the dummy variable.

In [2], motivated by (1.1) and (1.2), together with

$$\sum_{k=1}^{n} F_k^2 F_{k+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}, \tag{1.3}$$

we obtained several families of similar sums which involve longer products. For example, we obtained

$$\sum_{k=1}^{n} F_k F_{k+1} \ldots F_{k+2m}^2 \ldots F_{k+4m} = \frac{F_n F_{n+1} \ldots F_{n+4m+1}}{L_{2m+1}}, \tag{1.4}$$

for m a positive integer. By introducing a second parameter, s, we have managed to generalize all of the results in [2], while maintaining their elegance. The object of this paper is to present these generalizations, together with several results involving alternating sums, the like of which were not treated in [2]. In Section 2 we state our results, and in Section 3 we indicate the method of proof. We require the following identities:

$$F_{n+k} + F_{n-k} = F_n L_k, \quad k \text{ even}, \tag{1.5}$$

$$F_{n+k} + F_{n-k} = L_n F_k, \quad k \text{ odd}, \tag{1.6}$$

$$F_{n+k} - F_{n-k} = F_n L_k, \quad k \text{ odd}, \tag{1.7}$$

$$F_{n+k} - F_{n-k} = L_n F_k, \quad k \text{ even}, \tag{1.8}$$

$$L_{n+k} + L_{n-k} = L_n L_k, \quad k \text{ even}, \tag{1.9}$$

$$L_{n+k} + L_{n-k} = 5F_n F_k, \quad k \text{ odd}, \tag{1.10}$$

$$L_{n+k} - L_{n-k} = L_n L_k, \quad k \text{ odd}, \tag{1.11}$$

$$L_{n+k} - L_{n-k} = 5F_n F_k, \quad k \text{ even}, \tag{1.12}$$

$$L_n^2 - L_{2n} = (-1)^n 2 = (-1)^n L_0, \tag{1.13}$$

$$5F_n^2 - L_{2n} = (-1)^{n+1} 2 = (-1)^{n+1} L_0, \tag{1.14}$$

$$5F_{2n}^2 - L_{2n}^2 = -4 = -L_0^2. \tag{1.15}$$

Identities (1.5)-(1.12) occur as (5)-(12) in Bergum and Hoggatt [1], while (1.13)-(1.15) can be proved with the use of the Binet forms. In some of the proofs we need to recall the well-known identity $F_{2n} = F_n L_n$.

2. THE RESULTS

In this section we list our results in eight theorems, in which $s > 0$ and $m \geq 0$ are integers. In some of the theorems the parity of s is important, and the reasons for this become apparent in Section 3. Our numbering of Theorems 1-5 parallels that in [2], so that both sets of results can be easily compared.

Theorem 1:

$$\sum_{k=1}^{n} F_{sk} F_{s(k+1)} \cdots F_{s(k+4m)} L_{s(k+2m)} = \frac{F_{sn} F_{s(n+1)} \cdots F_{s(n+4m+1)}}{F_{s(2m+1)}}, \quad s \text{ even}, \tag{2.1}$$

$$\sum_{k=1}^{n} F_{sk} \cdots F_{s(k+2m)}^2 \cdots F_{s(k+4m)} = \frac{F_{sn} F_{s(n+1)} \cdots F_{s(n+4m+1)}}{L_{s(2m+1)}}, \quad s \text{ odd}. \tag{2.2}$$

Theorem 2:

$$\sum_{k=1}^{n} L_{sk} L_{s(k+1)} \cdots L_{s(k+4m)} F_{s(k+2m)} = \left[\frac{L_{sj} L_{s(j+1)} \cdots L_{s(j+4m+1)}}{5F_{s(2m+1)}} \right]_0^n, \quad s \text{ even}, \tag{2.3}$$

$$\sum_{k=1}^{n} L_{sk} L_{s(k+1)} \cdots L_{s(k+2m)}^2 \cdots L_{s(k+4m)} = \left[\frac{L_{sj} L_{s(j+1)} \cdots L_{s(j+4m+1)}}{L_{s(2m+1)}} \right]_0^n, \quad s \text{ odd}. \tag{2.4}$$

Theorem 3:

$$\sum_{k=1}^{n} F_{sk} F_{s(k+1)} \cdots F_{s(k+4m+2)} L_{s(k+2m+1)} = \frac{F_{sn} F_{s(n+1)} \cdots F_{s(n+4m+3)}}{F_{s(2m+2)}}, \tag{2.5}$$

$$\sum_{k=1}^{n} L_{sk} L_{s(k+1)} \cdots L_{s(k+4m+2)} F_{s(k+2m+1)} = \left[\frac{L_{sj} L_{s(j+1)} \cdots L_{s(j+4m+3)}}{5F_{s(2m+2)}} \right]_0^n. \tag{2.6}$$

Theorem 4:

$$\sum_{k=1}^{n} F_{sk}^2 F_{s(k+1)}^2 \cdots F_{s(k+4m)}^2 F_{s(2k+4m)} = \frac{F_{sn}^2 F_{s(n+1)}^2 \cdots F_{s(n+4m+1)}^2}{F_{s(4m+2)}}, \tag{2.7}$$

$$\sum_{k=1}^{n} L_{sk}^2 L_{s(k+1)}^2 \cdots L_{s(k+4m)}^2 F_{s(2k+4m)} = \left[\frac{L_{sj}^2 L_{s(j+1)}^2 \cdots L_{s(j+4m+1)}^2}{5F_{s(4m+2)}} \right]_0^n. \tag{2.8}$$

Theorem 5:

$$\sum_{k=1}^{n} F_{sk}^2 F_{s(k+1)}^2 \cdots F_{s(k+4m+2)}^2 F_{s(2k+4m+2)} = \frac{F_{sn}^2 F_{s(n+1)}^2 \cdots F_{s(n+4m+3)}^2}{F_{s(4m+4)}}, \tag{2.9}$$

$$\sum_{k=1}^{n} L_{sk}^2 L_{s(k+1)}^2 \cdots L_{s(k+4m+2)}^2 F_{s(2k+4m+2)} = \left[\frac{L_{sj}^2 L_{s(j+1)}^2 \cdots L_{s(j+4m+3)}^2}{5F_{s(4m+4)}}\right]_0^n. \tag{2.10}$$

For $m=0$ we interpret the summands in (2.2) and (2.4) as F_{sk}^2 and L_{sk}^2, respectively. For s odd the corresponding sums are then

$$\sum_{k=1}^{n} F_{sk}^2 = \frac{F_{sn}F_{s(n+1)}}{L_s} \quad \text{and} \quad \sum_{k=1}^{n} L_{sk}^2 = \left[\frac{L_{sj}L_{s(j+1)}}{L_s}\right]_0^n, \tag{2.11}$$

which generalize (1.1) and (1.2), respectively.

Interestingly, for $m=0$, (2.1) and (2.3) provide alternative expressions for the same sum, namely,

$$\sum_{k=1}^{n} F_{2sk} = \frac{F_{sn}F_{s(n+1)}}{F_s} = \left[\frac{L_{sj}L_{s(j+1)}}{5F_s}\right]_0^n, \quad s \text{ even}. \tag{2.12}$$

Theorem 6:

$$\sum_{k=1}^{n} (-1)^k F_{sk}F_{s(k+1)} \cdots F_{s(k+4m)}F_{s(k+2m)} = \frac{(-1)^n F_{sn}F_{s(n+1)} \cdots F_{s(n+4m+1)}}{L_{s(2m+1)}}, \quad s \text{ even}, \tag{2.13}$$

$$\sum_{k=1}^{n} (-1)^k F_{sk}F_{s(k+1)} \cdots F_{s(k+4m)}L_{s(k+2m)} = \frac{(-1)^n F_{sn}F_{s(n+1)} \cdots F_{s(n+4m+1)}}{F_{s(2m+1)}}, \quad s \text{ odd}. \tag{2.14}$$

Theorem 7:

$$\sum_{k=1}^{n} (-1)^k L_{sk}L_{s(k+1)} \cdots L_{s(k+4m)}L_{s(k+2m)} = \left[\frac{(-1)^n L_{sj}L_{s(j+1)} \cdots L_{s(j+4m+1)}}{L_{s(2m+1)}}\right]_0^n, \quad s \text{ even}, \tag{2.15}$$

$$\sum_{k=1}^{n} (-1)^k L_{sk}L_{s(k+1)} \cdots L_{s(k+4m)}F_{s(k+2m)} = \left[\frac{(-1)^n L_{sj}L_{s(j+1)} \cdots L_{s(j+4m+1)}}{5F_{s(2m+1)}}\right]_0^n, \quad s \text{ odd}. \tag{2.16}$$

Theorem 8:

$$\sum_{k=1}^{n} (-1)^k F_{sk}F_{s(k+1)} \cdots F_{s(k+4m+2)}F_{s(k+2m+1)} = \frac{(-1)^n F_{sn}F_{s(n+1)} \cdots F_{s(n+4m+3)}}{L_{s(2m+2)}}, \tag{2.17}$$

$$\sum_{k=1}^{n} (-1)^k L_{sk}L_{s(k+1)} \cdots L_{s(k+4m+2)}L_{s(k+2m+1)} = \left[\frac{(-1)^n L_{sj}L_{s(j+1)} \cdots L_{s(j+4m+3)}}{L_{s(2m+2)}}\right]_0^n. \tag{2.18}$$

Some special cases of these alternating sums are worthy of note. For $m=0$ Theorem 6 yields

$$\sum_{k=1}^{n}(-1)^k F_{sk}^2 = \frac{(-1)^n F_{sn}F_{s(n+1)}}{L_s}, \quad s \text{ even}, \tag{2.19}$$

and

$$\sum_{k=1}^{n}(-1)^k F_{2sk} = \frac{(-1)^n F_{sn}F_{s(n+1)}}{F_s}, \quad s \text{ odd}. \tag{2.20}$$

An alternative formulation for (2.20) is provided by (2.16). For $m = 0$ (2.15) becomes

$$\sum_{k=1}^{n}(-1)^k L_{sk}^2 = \left[\frac{(-1)^n L_{sj}L_{s(j+1)}}{L_s}\right]_0^n, \quad s \text{ even}. \tag{2.21}$$

3. THE METHOD OF PROOF

Each result in Section 2 can be proved with the use of the method in [2]. However, the significance of the parity of s in some of our theorems becomes apparent only when we work through the proofs. For this reason, we illustrate the method of proof once more by proving (2.4).

Proof of (2.4): Let l_n denote the sum on the left side of (2.4) and let

$$r_n = \frac{L_{sn}L_{s(n+1)}\cdots L_{s(n+4m+1)}}{L_{s(2m+1)}}.$$

Then

$$\begin{aligned}
r_n - r_{n-1} &= \frac{L_{sn}L_{s(n+1)}\cdots L_{s(n+4m)}}{L_{s(2m+1)}}[L_{s(n+4m+1)} - L_{s(n-1)}] \\
&= \frac{L_{sn}L_{s(n+1)}\cdots L_{s(n+4m)}}{L_{s(2m+1)}}[L_{s(n+2m)+s(2m+1)} - L_{s(n+2m)-s(2m+1)}] \\
&= L_{sn}L_{s(n+1)}\cdots L_{s(n+2m)}^2\cdots L_{s(n+4m)} \quad \text{[by (1.11) since } s(2m+1) \text{ is odd]} \\
&= l_n - l_{n-1}.
\end{aligned}$$

Thus $l_n - r_n = c$, where c is a constant.

Now

$$\begin{aligned}
c &= l_1 - r_1 \\
&= L_s L_{2s}\cdots L_{s(4m+1)}\left[L_{s(2m+1)} - \frac{L_{s(4m+2)}}{L_{s(2m+1)}}\right] \\
&= L_s L_{2s}\cdots L_{s(4m+1)}\cdot\frac{L_{s(2m+1)}^2 - L_{s(4m+2)}}{L_{s(2m+1)}} \\
&= -\frac{L_0 L_s L_{2s}\cdots L_{s(4m+1)}}{L_{s(2m+1)}} \quad \text{[by (1.13)]} \\
&= -r_0,
\end{aligned}$$

and this concludes the proof. □

In contrast, when proving (2.3), we are required to factorize $L_{s(n+2m)+s(2m+1)} - L_{s(n+2m)-s(2m+1)}$ for s even, and this requires the use of (1.12).

As in [2], we conclude by mentioning that the results of this paper translate immediately to the sequences defined by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \quad U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \quad V_1 = p. \end{cases}$$

We simply replace F_n by U_n, L_n by V_n, and 5 by p^2+4.

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REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences." *The Fibonacci Quarterly* **13.2** (1975):115-20.
2. R. S. Melham. "Sums of Certain Products of Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **37.3** (1999):248-51

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