ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-561 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let n be an integer and set

$$s_{n+1}=\alpha^n+\alpha^{n-1}\beta+\cdots+\alpha\beta^{n-1}+\beta^n,$$

where $\alpha+\beta=a$, $\alpha\beta=b$, with $a\neq 0$, $b\neq 0$ two arbitrary parameters. Then prove that:

a) $s_p^r s_{qr+n}=\sum_{\ell=0}^{r}\binom{r}{\ell}b^{q(r-\ell)}s_q^\ell s_{p-q}^{r-\ell}s_{p\ell+n};$

b) $b^{pr}s_q^r s_n=\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}s_p^\ell s_{q+p}^{r-\ell}s_{q\ell+pr+n};$

c) $s_{2p+q}^r s_{qr+n}=\sum_{\ell=0}^{r}\binom{r}{\ell}b^{(p+q)(r-\ell)}s_p^{r-\ell}s_{p+q}^\ell s_{(2p+q)\ell-pr+n};$

where $r\geq 0$, n, $p(\neq 0)$, and $q(\neq 0,\pm p)$ are arbitrary integers.

H-562 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that, for all nonnegative integers n,

$$L_{2n+1}=4^n-5\sum_{k=0}^{[\frac{n-2}{5}]}\binom{2n+1}{n-5k-2},$$

where [ ] denotes the greatest integer function.

H-563 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let $m>0$, $n\geq 0$, $p\neq 0$, $q\neq -p, 0$, and s be integers and, for $1\leq k\leq n$, let $(n)_k:=n(n-1)\ldots(n-k+1)$ and $S_m^{(k)}$ be a Stirling number of the second kind.

Prove the following identity for Fibonacci numbers:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}r^m[F_p/F_{p+q}]^rF_{qr+s}$$

$$=(-1)^{np}[F_q/F_{p+q}]^n\sum_{k=1}^{m}(-1)^{(p+1)k}(n)_kS_m^{(k)}[F_p/F_q]^kF_{(p+q)k-np+s}.$$

SOLUTIONS

An Odd Problem

H-545 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 36, no. 5, November 1998)

Prove that, for all odd primes p,

(a) $\sum_{k=1}^{p-1} L_k \cdot k^{-1} \equiv \frac{-2}{p}(L_p - 1) \pmod{p}$;

(b) $\sum_{k=1}^{p-1} F_k \cdot k^{-1} \equiv 0 \pmod{p}$.

Solution by the proposer

We first observe that $L_p \equiv 1 \pmod{p}$ for all primes p; thus, all the expressions indicated in (a) and (b) are well-defined integers (mod p). Now

$$\alpha^p = (1-\beta)^p = \sum_{k=0}^{p} \binom{p}{k} (-\beta)^k = 1 - \beta^p + \sum_{k=1}^{p-1} \frac{p}{k} \binom{p-1}{k-1} (-\beta)^k.$$

Hence,

$$\frac{1}{p}(L_p - 1) = \sum_{k=1}^{p-1} \frac{1}{k} \binom{p-1}{k-1} (-\beta)^k.$$

Now

$$\binom{p-1}{k-1} \equiv \binom{-1}{k-1} = (-1)^{k-1} \pmod{p}.$$

Thus,

$$\frac{1}{p}(L_p - 1) \equiv -\sum_{k=1}^{p-1} k^{-1} \cdot \beta^k \pmod{p}.$$

Similarly, it is also true that

$$\frac{1}{p}(L_p - 1) \equiv -\sum_{k=1}^{p-1} k^{-1} \cdot \alpha^k \pmod{p}.$$

Adding and subtracting the last two congruences yields (a) and (b), respectively.

Note: From (a) and (b), it follows that a necessary and sufficient condition for $p^2 \mid (L_p - 1)$ is that

$$\sum_{k=1}^{p-1} F_{k+n} \cdot k^{-1} \equiv 0 \pmod{p}, \text{ for all integers } n.$$

Equivalently,

$$\sum_{k=1}^{p-1} L_{k+n} \cdot k^{-1} \equiv 0 \pmod{p}, \text{ for all } n.$$

Other equivalent forms of such conditions are:

$$\sum_{k=1}^{p-1} F_k \cdot k^{-1} \equiv \sum_{k=1}^{p-1} F_{k+1} \cdot k^{-1} \equiv 0 \pmod{p}$$

or

$$\sum_{k=1}^{p-1} L_k \cdot k^{-1} \equiv \sum_{k=1}^{p-1} L_{k+1} \cdot k^{-1} \equiv 0 \pmod{p}.$$

In turn, these conditions are equivalent to the condition that $Z(p^2) = Z(p)$, where $Z(m)$ is the "Fibonacci entry-point" of m (i.e., the smallest positive integer n such that $m \mid F_n$).

Also solved by H.-J. Seiffert.

A Strange Triangle

H-546 ***Proposed by André-Jeannin, Longwy, France (Vol. 36, no. 5, November, 1998)***

Find the triangular Mersenne numbers. (The sequence of Mersenne numbers is defined by $M_n = 2^n - 1$.)

Solution by the proposer

We shall prove that the only Mersenne triangular numbers are M_0, M_1, M_2, M_4, and M_{12}. In fact, the equation

$$M_n = 2^n - 1 = \frac{k(k+1)}{2}$$

is clearly equivalent to the equation

$$x^2 = 2^{n+3} - 7, \tag{1}$$

where $x = 2k+1$.

It is known [1] that (1) admits the only positive solutions $(n = 0, x = 1)$, $(n = 1, x = 3)$, $(n = 2, x = 5)$, $(n = 4, x = 11)$, and $(n = 12, x = 181)$. The result follows.

Reference

1. Th. Skolem, P. Chowla, & D. J. Lewis. "The Diophantine Equation $2^{n+2} - 7 = x^2$ and Related Problems." *Proc. Amer. Math. Soc.* **10** (1959):663-69.

Also solved by P. Bruckman and H.-J. Seiffert.

A Prime Problem

H-547 ***Proposed by T. V. Padmakumar, Thycaud, India (Vol. 37, no. 1, February 1999)***

If p is a prime number, then

$$\left[\sum_{n=1}^{p} \frac{1}{(2n-1)}\right]^2 - \left[\sum_{n=1}^{p} \frac{1}{(2n-1)^2}\right] \equiv 0 \pmod{p}.$$

Solution by L. A. G. Dresel, Reading, England

Note: The result is clearly true for $p = 2$. However, when p is an odd prime, each summation contains the undefined term p^{-1} (mod p). Therefore, we shall assume that these terms are to be omitted (or, possibly, consider then as *formally* canceling each other). The result is then true for $p \geq 5$ but false for $p = 3$.

Proof for $p \geq 5$: For $1 \leq n \leq p$, the sequence of odd numbers $2n-1$ (mod p) reproduces the residues 0, 1, 2, ..., $p-1$ in a different order. Omitting the residue 0, as explained above, consider the summations over $1 \leq s \leq p-1$, $A \equiv \sum s^{-1}$, and $B \equiv \sum s^{-2}$ (mod p).

Now consider the reciprocals modulo p of two residues s and t. Then it is easily shown that $s^{-1} \equiv t^{-1}$ (mod p) if and only if $s \equiv t$ (mod p). Hence, all the terms in the summation A are distinct (mod p), so that we have $A \equiv \sum s$ and, similarly, we obtain $B \equiv \sum s^2$ (mod p).

Finally consider the equation $x^{p-1} - 1 \equiv 0$ (mod p). By Fermat's theorem, this is satisfied for $x = 1, 2, \ldots, p-1$, so we can write $x^{p-1} - 1 \equiv (x-1)(x-2)\ldots(x-p+1)$ (mod p). If $p > 2$, it follows that the sum of the roots is zero, and if $p > 3$, we also have the sum of the products of the roots taken two at a time is zero (mod p). Hence, we have $A \equiv 0$, and also $A^2 - B \equiv 0$ (mod p) for $p \geq 5$.

Also solved by P. Bruckman, H. Kwong, and the proposer.

Pell-Mell

H-548 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 37, no. 1, February 1999)***

Define the sequence of Pell numbers by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Show that, if q is a prime such that $q \equiv 1$ (mod 8), then

$$q \mid P_{(q-1)/4} \text{ if and only if } 2^{(q-1)/4} \equiv (-1)^{(q-1)/8} \pmod{q}.$$

Solution by the proposer

Consider the Lucas polynomials defined by $L_0(x) = 2$, $L_1(x) = x$, and $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$ for $n \geq 0$. It is well known that

$$L_n(x) = \left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n, \quad n \geq 0. \tag{1}$$

Let $Q_n = L_n(2)$, $n \geq 0$, denote the n^{th} Pell-Lucas number.

Proposition: For all $n \geq 0$, it holds that

$$Q_n = 2^{-[(n-1)/2]} \sum_{\substack{k=0 \\ 4 \nmid n+2k+2}}^{n} (-1)^{[(n-2k+1)/4]} \binom{2n}{2k},$$

where [ ] denotes the greatest integer function.

Proof: If $t \neq 1$ is any complex number, then by (1),

$$L_n\left(2i\frac{1+t}{1-t}\right) = \frac{i^n}{(1-t)^n}\left((1+\sqrt{t})^{2n} + (1-\sqrt{t})^{2n}\right),$$

where $i = \sqrt{(-1)}$. Applying the binomial theorem gives

$$L_n\left(2i\frac{1+t}{1-t}\right) = \frac{2i^n}{(1-t)^n}\sum_{k=0}^{n}\binom{2n}{2k}t^k.$$

Now we take $t=-i$. Since $(1-i)/(1+i)=-i$, $1/(i+1)=(1-i)/2$, $-i=1/i$, and $L_n(2)=Q_n$, we find

$$Q_n = 2^{1-n}\sum_{k=0}^{n}\binom{2n}{2k}i^{n-k}(1-i)^n.$$

Using $i=e^{i\pi/2}$ and $1-i=\sqrt{2}\,e^{-i\pi/4}$ yields

$$Q_n = 2^{1-n/2}\sum_{k=0}^{n}\binom{2n}{2k}\exp\left(i(n-2k)\frac{\pi}{4}\right).$$

Equating the real parts gives

$$Q_n = 2^{1-n/2}\sum_{k=0}^{n}\binom{2n}{2k}A_{n-2k},$$

where $A_j := \cos(j\pi/4)$, $j\in Z$. An elementary calculation shows that

$$A_j = \begin{cases}(-1)^{[(j+1)/4]}2^{[j/2]-j/2} & \text{if } j\not\equiv 2 \pmod 4,\\ 0 & \text{if } j\equiv 2 \pmod 4.\end{cases}$$

The stated identity easily follows. Q.E.D.

The next result is known.

Lemma: If q is a prime, then

$$\binom{q-1}{k}\equiv(-1)^k \pmod q \text{ for } k=1,\ldots,q-1.$$

Proof: Since q is a prime, q divides $\binom{q}{k}$ for $k=1,\ldots,q-1$. Hence, the equation

$$\binom{q}{k}=\binom{q-1}{k}+\binom{q-1}{k-1}$$

implies that

$$\binom{q-1}{k}\equiv-\binom{q-1}{k-1} \pmod q \text{ for } k=1,\ldots,q-1,$$

so that the desired congruence can be proved by a simple induction argument. Q.E.D.

If q is a prime such that $q\equiv 1 \pmod 8$, then $q=8j+1$ for some positive integer j. Using the identity of the proposition with $n=(q-1)/2$ and applying the Lemma, modulo q we find that

$$2^{(q-5)/4}Q_{(q-1)/2}\equiv\sum_{\substack{k=0\\ k \text{ even}}}^{4j}(-1)^{[j-(2k-1)/4]}=\sum_{r=0}^{2j}(-1)^{j-r}=(-1)^j \pmod q$$

or

$$2^{(q-5)/4}Q_{(q-1)/2}\equiv(-1)^{(q-1)/8} \pmod q. \tag{2}$$

The well-known identity $8P_n^2=Q_{2n}-2(-1)^n$ with $n=(q-1)/4$ and (2) imply that

$$2^{(q+7)/4}P_{(q-1)/4}^2\equiv(-1)^{(q-1)/8}-2^{(q-1)/4} \pmod q.$$

This proves the desired criterion.

Remark: The two smallest such primes are $q=41$ and $q=113$. In fact, we have $P_{10}=2378=41\cdot 58$ and $P_{28}=18457556052=113\cdot 163341204$.

Also solved by P. Bruckman

Resurrection

H-549 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 1, February 1999)

Evaluate the expression: $\sum_{n\geq 1}(-1)^{n-1}\tan^{-1}(1/F_{2n})$. (1)

Note: A number of readers have pointed out that this problem appeared in the *Quarterly* (Vol. 1, no. 4, 1963) on page 71 as Theorem 5.

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

Note first that this problem was presented as a theorem by Hoggatt and Ruggles in [2].

Lemma 1: $\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right)-\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right)=\tan^{-1}\left(\frac{(-1)^{n-1}}{F_{n+2}}\right)$.

Proof: Using (I_{10}), $F_{2n}=F_{n+1}^2-F_{n-1}^2$, and (I_{13}), $F_{n-1}F_{n+1}-F_n^2=(-1)^n$, see Hoggatt [1],

$$\frac{(-1)^{n-1}}{F_{2n+2}}=\frac{(-1)^{n-1}}{F_{(n+1)+1}^2-F_{(n+1)-1}^2}=\frac{(-1)^{n-1}}{F_{n+2}^2-F_n^2}=\frac{(-1)^{n-1}}{(F_{n+2}-F_n)(F_{n+2}+F_n)}$$

$$=\frac{F_nF_{n+2}-F_{n+1}^2}{F_{n+1}(F_{n+2}+F_n)}=\frac{\frac{F_n}{F_{n+1}}-\frac{F_{n+1}}{F_{n+2}}}{1+\frac{F_n}{F_{n+2}}}=\frac{\frac{F_n}{F_{n+1}}-\frac{F_{n+1}}{F_{n+2}}}{1+\left(\frac{F_n}{F_{n+1}}\right)\left(\frac{F_{n+1}}{F_{n+2}}\right)}$$

$$=\tan\left(\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right)-\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right)\right).$$

The lemma follows by taking inverse tangents.

Lemma 2: $\sum_{m=1}^{n}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m}}\right)=\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right)$.

Proof: Using Lemma 1, it is seen that the series telescopes:

$$\sum_{m=1}^{n}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m}}\right)=\tan^{-1}\frac{1}{F_2}-\tan^{-1}\frac{1}{F_4}+\tan^{-1}\frac{1}{F_6}-\tan^{-1}\frac{1}{F_8}+\cdots+(-1)^{m-1}\tan^{-1}\frac{1}{F_{2m}}$$

$$=\tan^{-1}\frac{F_1}{F_2}-\tan^{-1}\frac{F_0}{F_1}+\tan^{-1}\frac{F_2}{F_3}-\tan^{-1}\frac{F_1}{F_2}+\tan^{-1}\frac{F_3}{F_4}-\tan^{-1}\frac{F_2}{F_3}$$

$$+\tan^{-1}\frac{F_4}{F_5}-\tan^{-1}\frac{F_3}{F_4}+\cdots+\tan^{-1}\frac{F_n}{F_{n+1}}-\tan^{-1}\frac{F_{n-1}}{F_n}$$

$$=\tan^{-1}\frac{F_n}{F_{n+1}}-\tan^{-1}\frac{F_0}{F_1}=\tan^{-1}\frac{F_n}{F_{n+1}}.$$

This completes the proof of Lemma 2.

Note that the arctangent function is continuous and increasing on the interval (0, 1), so

$$\tan^{-1}\left(\frac{1}{F_{2n+2}}\right) \le \tan^{-1}\left(\frac{1}{F_{2n}}\right)$$

and that the series

$$\sum_{n=1}^{\infty}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2n}}\right)$$

is alternating with $\lim_{n\to\infty}(1/F_{2n}) = 0$, and thus converges to some value A, say. Therefore,

$$\sum_{m=1}^{n}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m}}\right) \le A \le \sum_{m=1}^{n+1}(-1)^{m-1}\tan^{-1}\left(\frac{1}{F_{2m+2}}\right)$$

for n an odd integer. So, by Lemma 2,

$$\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right) \le A \le \tan^{-1}\left(\frac{F_n}{F_{n+1}}\right).$$

Taking limits and using the well-known result that $\lim_{n\to\infty}(F_n/F_{n+1}) = \frac{1}{\alpha} = (\sqrt{5}-1)/2$, the golden number (see Hoggatt, [3]), it follows that

$$\lim_{n\to\infty}\tan^{-1}\left(\frac{F_{n+1}}{F_{n+2}}\right) \le A \le \lim_{n\to\infty}\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right) \Rightarrow \tan^{-1}\frac{1}{\alpha} \le A \le \tan^{-1}\frac{1}{\alpha}.$$

Thus,

$$A = \tan^{-1}\left(\frac{2}{\sqrt{5}+1}\right) = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

A similar argument works for the case in which n is an even integer. In either case, the value of the given expression is

$$\sum_{n\ge 1}(-1)^{n-1}\tan^{-1}\left(\frac{1}{F_{2n}}\right) = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

References

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
2. V. E. Hoggatt, Jr., & I. D. Ruggles. "A Primer for the Fibonacci Numbers–Part IV." *The Fibonacci Quarterly* **1.4** (1963):65-71.
3. V. E. Hoggatt, Jr., & I. D. Ruggles. "A Primer for the Fibonacci Numbers–Part V." *The Fibonacci Quarterly* **2.1** (1964):61.

Also solved by P. Bruckman, L. A. G. Dresel, H. Kwong, H.-J. Seiffert, and I. Strazdins.

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