# A GEOMETRIC CONNECTION BETWEEN GENERALIZED FIBONACCI SEQUENCES AND NEARLY GOLDEN SECTIONS

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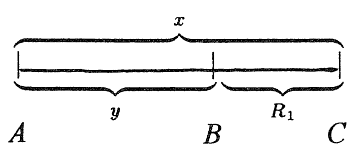
*(Submitted June 1998-Final Revision November 1999)*

## 1. INTRODUCTION

We introduce a way to construct generalized golden sections, and demonstrate a geometric connection between these sections and generalized Fibonacci sequences of the form $u_{n+1} = k \cdot u_n + u_{n-1}$, where $u_0 = 0$, $u_1 = 1$. for $k \geq 1$. We let $\phi = (1+\sqrt{5})/2$, the golden ratio, and $F_n^{(k)}$ represent the $n^{\text{th}}$ term of the $k^{\text{th}}$ generalized Fibonacci sequence, defined above. Our method will employ a geometric version of the Euclidean Algorithm.

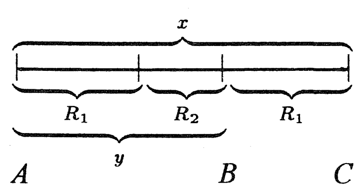
For $k = 1$, the key fact is that if two line segments with lengths $x$ and $y$ satisfy $x/y = \phi$, then $x = y + R_1$, where $R_1 < y$ and $y/R_1$ is itself equal to $\phi$. This follows from the definition of the golden section. See Figure 1 and the mathematical argument given in [3, pp. 9-10].




$$\frac{y}{R_1} = \frac{AB}{BC} = \frac{AC}{AB} = \frac{x}{y} = \phi$$

**FIGURE 1**

Since $x = y + R_1$, and $R_1 < y$, we can approximate $x$ (badly) by ignoring the remainder $R_1$, and estimate $x/y = (y+R_1)/y \approx 1$. To refine this estimate, we should use a smaller unit with which to measure. Hence, we now choose $R_1$. This is shown in Figure 2.



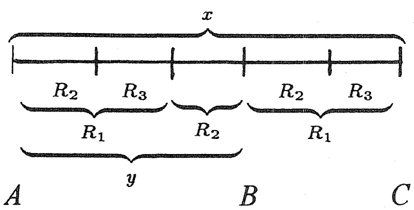

$$\frac{R_1}{R_2} = \frac{y}{R_1} = \frac{x}{y} = \phi$$

**FIGURE 2**

From Figure 2, a new estimate of $x/y$, ignoring the remainder $R_2$, is

$$\frac{x}{y} = \frac{2R_1 + R_2}{R_1 + R_2} \approx 2.$$

If we now lay off $R_2$ against each $R_1$, we have the construct in Figure 3.




$$\frac{R_2}{R_3} = \frac{R_1}{R_2} = \frac{y}{R_1} = \frac{x}{y} = \phi$$

**FIGURE 3**

From Figure 3, a new estimate, this time ignoring the remainder $R_3$, is

$$\frac{x}{y} = \frac{3R_2 + 2R_3}{2R_2 + R_3} \approx 3/2.$$

If we continue this process, it is easy to see and to prove by induction that

$$\phi = \frac{x}{y} = \frac{F_{n+2} \cdot R_n + F_{n+1} \cdot R_{n+1}}{F_{n+1} \cdot R_n + F_n \cdot R_{n+1}} \approx \frac{F_{n+2}}{F_{n+1}}.$$

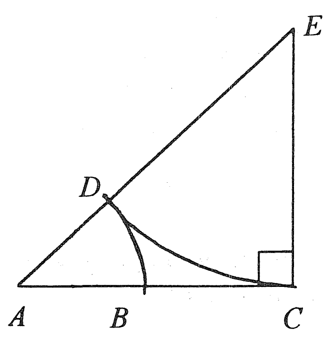
This gives a geometric flavor to the well-known identity

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi.$$

## 2. NEARLY GOLDEN SECTIONS

We generalize the golden section in a manner not entirely unlike Philip Engstrom's generalization in [2]. We do so by giving a ruler and compass method of locating point $B$, between $A$ and $C$, as in Figure 4 below.

Let $k$ be a fixed positive integer. Given a line segment $\overline{AC}$, first bisect the segment. Construct a perpendicular $\overline{EC}$ at point $C$ of length $\frac{k}{2} \cdot AC$. By striking arcs, locate points $D$ and $B$, as shown in Figure 4, so that $DE = CE$ and $AB = AD$.




$$\overline{EC} \perp \overline{AC}$$
$$EC = \tfrac{k}{2} \cdot AC$$
$$DE = CE,\ AB = AD$$

**FIGURE 4**

By the Pythagorean Theorem,

$$AE = \sqrt{(AC)^2 + (EC)^2} = \sqrt{(AC)^2 + \frac{k^2}{4}(AC)^2} = \frac{1}{2}\sqrt{k^2+4} \cdot AC.$$

So we have

$$\begin{aligned} AB &= AD = AE - DE = AE - CE \\ &= \frac{1}{2}\sqrt{k^2+4} \cdot AC - \frac{k}{2} \cdot AC = \frac{-k+\sqrt{k^2+4}}{2} \cdot AC. \end{aligned}$$

It follows that

$$\frac{AC}{AB} = \frac{2}{-k+\sqrt{k^2+4}} = \frac{k+\sqrt{k^2+4}}{2}.$$

We shall call this ratio $\phi_k$, the $k^{\text{th}}$ *generalized golden ratio*. That is,

$$\phi_k = \frac{k+\sqrt{k^2+4}}{2}$$

Letting $t_k = -1/\phi_k$, the other root of the equation $t^2 - kt - 1 = 0$, it is now a simple exercise to follow the reasoning in Hoggatt's book [3, pp. 10-11], to establish the Binet form

$$F_n^{(k)} = \frac{\phi_k^n - t_k^n}{\phi_k - t_k}.$$

Using the notation of Horadam [4, p. 161], $F_n^{(k)} = w(0, 1; k, -1)$, a generalized Fibonacci sequence of the form mentioned in the introduction. In [4] and a large number of other articles appearing in this journal, one can find many formulas for the sequences $F_n^{(k)}$ and the related generalized Lucas sequences given by $L_n^{(k)} = \phi_k^n + t_k^n$. However, one formula we did not find is $(\phi_{2m+1} - m)^2 = (m^2+1) + \phi_{2m+1}$. This formula is easy to prove by using the formula for the value of $\phi_k$ given above. This identity implies that, for odd $k$, the decimal part of $\phi_k$ is the decimal part of a number which differs from its square by a positive integer. The table below gives some examples to illustrate this.

**TABLE 1. A Squaring Property**

| $m$ | $\phi_{2m+1} - m$ | $(\phi_{2m+1} - m)^2$ |
|---|---|---|
| 0 | 1.6180339887... | 2.6180339887... |
| 1 | 2.3027756377... | 5.3027756377... |
| 2 | 3.1925824036... | 10.1925824036... |
| 3 | 4.1400549446... | 17.1400549446... |

## 3. THE GEOMETRIC CONNECTION FOR GENERALIZED FIBONACCI SEQUENCES

We now use the construction of Section 2 to emulate the geometric process of Section 1 for approximating $\phi_k$ for $k \geq 2$. The goal is to demonstrate a geometric connection, similar to the one shown in Section 1, between ratios of generalized Fibonacci numbers and generalized golden ratios.

***Definition:*** To form the $k^{\text{th}}$ *nearly golden section*, cut a line segment into $k+1$ pieces such that

*1.* $k$ of the pieces have equal length,
*2.* the remaining piece is shorter than the first $k$ pieces, and
*3.* the ratio of the length of a single larger piece to the smaller piece is equal to the length of the whole segment to that of the larger piece.

The construction of Section 2 tells us how to cut a line segment in this way. A few comments are in order.

With lengths as described in Figure 4, we have

$$\frac{AC-k\cdot AB}{AB}=\frac{AC}{AB}-k=\frac{-k+\sqrt{k^2+4}}{2},$$

and so,

$$\frac{AB}{AC-k\cdot AB}=\frac{2}{-k+\sqrt{k^2+4}}=\frac{k+\sqrt{k^2+4}}{2}=\frac{AC}{AB}=\phi_k.$$

From this calculation, we deduce first that when $AC/AB=\phi_k$, as in the construction, then $k\cdot AB<AC$. So, by duplicating the length $AB$ an additional $k-1$ times on the segment $\overline{AC}$, beginning at point $B$, we can cut the line segment $\overline{AC}$ in the manner illustrated for $k=2$ and $k=3$ below. (These are generalizations of the cut made in Figure 1.)

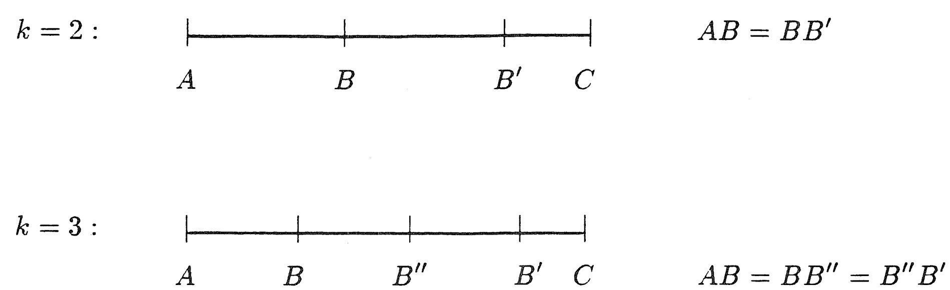



**FIGURE 5**

Said another way, if $A$, $B$, and $C$ are as in Figure 4, with $AC/AB=\phi_k$, then $AC=k\cdot AB+B'C$ (as in Figure 5). Moreover,

$$\frac{AB}{B'C}=\frac{AC}{AB}=\phi_k.$$

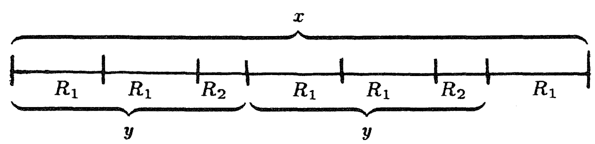
These facts allow us to emulate the geometric process we described in the introduction.

The key fact now, obtained from the preceding discussion, is that if two line segments with lengths $x$ and $y$ satisfy $x/y=\phi_k$ then $x=k\cdot y+R_1$, where $R_1<y$ and $y/R_1=\phi_k$. (In the definition, $x$ is the length of the original segment, $y$ that of one of the larger pieces, and $R_1$ that of the shorter piece. In Figure 5, $x=AC$, $y=AB$, and $R_1=B'C=AC-k\cdot AB$.) Thus, geometrically, $y$ can be laid off $k$ times against $x$, with a remainder of length $R_1$, and the ratio $y/R_1$ is the same as the original ratio $x/y$. This means that now $R_1$ can be laid off $k$ times against *each* $y$, with remainder $R_2=y-k\cdot R_1$, and $R_1/R_2=y/R_1=x/y=\phi_k$. This process can be repeated indefinitely.

We now estimate $\phi_k$. Our first estimate (ignoring the remainder $R_1$) is

$$\phi_k = \frac{x}{y} = \frac{k \cdot y + R_1}{y} = \frac{F_2^{(k)} \cdot y + F_1^{(k)} \cdot R_1}{F_1^{(k)} \cdot y + F_0^{(k)}} \approx \frac{F_2^{(k)}}{F_1^{(k)}}.$$

Now, as we said above, $y / R_1$ is also equal to $\phi_k$. So $y = k \cdot R_1 + R_2$, where $R_2 < R_1$ and $R_1 / R_2 = y / R_1 = \phi_k$. We can lay off $R_1$ $k$ times against *each* $y$. We illustrate for $k = 2$ in Figure 6 below.




$$\frac{y}{R_1} = \frac{x}{y} = \phi_2$$

**FIGURE 6**

By substitution, we have

$$x = k \cdot y + R_1 = k(k \cdot R_1 + R_2) + R_1 = (k^2 + 1)R_1 + k \cdot R_2,$$
$$y = k \cdot R_1 + R_2.$$

Since $F_1^{(k)} = 1$, $F_2^{(k)} = k$, and $F_3^{(k)} = k^2 + 1$, we may write

$$x = F_3^{(k)} \cdot R_1 + F_2^{(k)} \cdot R_2,$$
$$y = F_2^{(k)} \cdot R_1 + F_1^{(k)} \cdot R_2.$$

So our second estimate, this time ignoring the remainder $R_2$, is

$$\phi_k = \frac{x}{y} = \frac{F_3^{(k)} \cdot R_1 + F_2^{(k)} \cdot R_2}{F_2^{(k)} \cdot R_1 + F_1^{(k)} \cdot R_2} \approx \frac{F_3^{(k)}}{F_2^{(k)}}.$$

These are the first steps of an iterative process in which, at each step, $R_n$ is laid off $k$ times against *each* $R_{n-1}$ (since $R_{n-1} = k \cdot R_n + R_{n+1}$), and

$$\frac{x}{y} = \frac{y}{R_1} = \frac{R_1}{R_2} = \cdots = \frac{R_n}{R_{n+1}}.$$

At the $n^{\text{th}}$ step we have

$$\begin{aligned} x &= a_{n+1} \cdot R_{n-1} + a_n \cdot R_n, \\ y &= a_n \cdot R_{n-1} + a_{n-1} \cdot R_n. \end{aligned} \tag{1}$$

By substitution into (1), since $R_{n-1} = k \cdot R_n + R_{n+1}$, we have

$$\begin{aligned} x &= a_{n+1}(k \cdot R_n + R_{n+1}) + a_n \cdot R_n \\ &= \underbrace{(k \cdot a_{n+1} + a_n)}_{a_{n+2}} R_n + a_{n+1} \cdot R_{n+1}, \end{aligned}$$

$$\begin{aligned} y &= a_n(k \cdot R_n + R_{n+1}) + a_{n-1} \cdot R_n \\ &= \underbrace{(k \cdot a_n + a_{n-1})}_{a_{n+1}} R_n + a_n \cdot R_{n+1}. \end{aligned}$$

We see that the sequence $a_n$ is defined by the rule $a_{n+2} = k \cdot a_{n+1} + a_n$ for all $n \geq 1$. That is, $a_n = F_n^{(k)}$, and

$$\phi_k = \frac{x}{y} = \frac{F_{n+2}^{(k)} \cdot R_n + F_{n+1}^{(k)} \cdot R_{n+1}}{F_{n+1}^{(k)} \cdot R_n + F_n^{(k)} \cdot R_{n+1}} \approx \frac{F_{n+2}^{(k)}}{F_{n+1}^{(k)}}.$$

This is the desired generalization of the geometric approximation in the introduction.

## ACKNOWLEDGMENT

The author appreciates the patience and advice of the anonymous referee whose comments and suggestions contributed largely to improving the form and presentation of this article.

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AMS Classification Number: 11B39

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## A MESSAGE OF GRATITUDE TO DR. STANLEY RABINOWITZ

The Editor, Editorial Board, and Board of Directors of The Fibonacci Association wish to express their deep gratitude to Dr. Stanley Rabinowitz for his excellent work as Editor of the Elementary Problems and Solutions section of *The Fibonacci Quarterly.* Our best wishes go with him as he retires from this position after nine years to devote full time to his publishing enterprise, MathPro Press.