

THE EIGENVECTORS OF A CERTAIN MATRIX OF BINOMIAL COEFFICIENTS

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(Submitted May 1998-Final Revision August 1998)

1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{aligned} U_n &= pU_{n-1} - qU_{n-2}, & U_0 &= 0, & U_1 &= 1, \\ V_n &= pV_{n-1} - qV_{n-2}, & V_0 &= 2, & V_1 &= p, \end{aligned}$$

where p and q are real numbers with $q(p^2 - 4q) \neq 0$. These sequences were studied originally by Lucas [4], and have subsequently been the subject of much attention.

The Binet forms of U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an n^{th} root of unity for any n .

For n greater than or equal to 1, let $S(n)$ be the $n \times n$ matrix defined by

$$S(n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & (-1)^{n-1} \binom{n-1}{n-1} q^{n-1} \\ & & & \cdots & \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & \cdots & \\ 0 & 0 & \binom{2}{2} q^2 & \cdots & \binom{n-1}{2} p^{n-3} q^2 \\ 0 & -\binom{1}{1} q & -\binom{2}{1} p q & \cdots & -\binom{n-1}{1} p^{n-2} q \\ \binom{0}{0} & \binom{1}{0} p & \binom{2}{0} p^2 & \cdots & \binom{n-1}{0} p^{n-1} \end{pmatrix}.$$

The element in its i^{th} row and j^{th} column is

$$(-q)^{n-i} \binom{j-1}{j+i-n-1} p^{i+j-n-1}.$$

The matrix $S(n)$ is the general term in a sequence of matrices $\{S(n)\}_{n=1}^{\infty}$, where the first few terms are

$$S(1) = (1), \quad S(2) = \begin{pmatrix} 0 & -q \\ 1 & p \end{pmatrix}, \quad \text{and} \quad S(3) = \begin{pmatrix} 0 & 0 & q^2 \\ 0 & -q & -2pq \\ 1 & p & p^2 \end{pmatrix}.$$

It can be proved by induction that

$$S^n(2) = \begin{pmatrix} -qU_{n-1} & -qU_n \\ U_n & U_{n+1} \end{pmatrix}$$

and

$$S^n(3) = \begin{pmatrix} q^2U_{n-1}^2 & q^2U_{n-1}U_n & q^2U_n^2 \\ -2qU_{n-1}U_n & -q(U_n^2 + U_{n-1}U_{n+1}) & -2qU_nU_{n+1} \\ U_n^2 & U_nU_{n+1} & U_{n+1}^2 \end{pmatrix},$$

with similar results for the higher-order matrices in the sequence $\{S(n)\}_{n=1}^\infty$. When $p = -q = 1$, $S(2)$ becomes essentially the Q -Matrix for the Fibonacci numbers. For applications of $S(3)$ and $S(4)$ to the derivation of certain infinite series, and for more background information on these matrices, see [6] and [7].

Carlitz [1] considered the matrix $S(n)^T$ for the special case $p = -q = 1$. Among other things, he found its eigenvalues and its characteristic polynomial, and stated that its eigenvectors were not evident.

Mahon and Horadam [5] worked with the matrix $S(n)$ for the case $q = -1$ and put forward a conjecture stating its characteristic polynomial. This conjecture was later proved by Duvall and Vaughan [3].

More recently, Cooper and Kennedy [2] considered the matrix $S(n)^T$ and proved a result of Jarden by generalizing the work of Carlitz [1]. If we translate their results to our matrix $S(n)$, they proved, among other things:

- (i) The eigenvalues of $S(n)$ are $\alpha^{n-1}, \alpha^{n-2}\beta, \alpha^{n-3}\beta^2, \dots, \alpha\beta^{n-2}, \beta^{n-1}$.
- (ii) The characteristic equation of $S(n)$ is

$$\sum_{k=0}^n (-1)^k q^{(k-1)k/2} \{n, k\} \lambda^{n-k} = 0,$$

where

$$\{n, k\} = \begin{cases} 1, & \text{for } k = 0, n, \\ \frac{\prod_{i=1}^n U_i}{\left(\prod_{i=1}^k U_i\right)\left(\prod_{i=1}^{n-k} U_i\right)}, & \text{for } 0 < k < n. \end{cases}$$

There remains the question of the eigenvectors of $S(n)$. The purpose of this paper is to answer that question.

2. EIGENVECTORS OF $S(n)$

Let $0 \leq k \leq n-1$ be a fixed integer,

$$f(x) = (x - \alpha)^k (x - \beta)^{n-1-k} = \sum_{r=0}^{n-1} v_r x^r,$$

and

$$\mathbf{v} = (v_0, v_1, \dots, v_{n-1})^T.$$

Lemma 1: Let $m \geq 0$. Then

$$f^{(m)}(x) = m! \frac{f(x)}{(x-\alpha)^m(x-\beta)^m} \sum_{j=0}^m \binom{k}{m-j} \binom{n-1-k}{j} (x-\alpha)^j (x-\beta)^{m-j}.$$

Proof: We will use Leibniz's formula for the m^{th} derivative of a product, i.e.,

$$\frac{d^m}{dx^m} g(x)h(x) = \sum_{j=0}^m \binom{m}{j} g^{(m-j)}(x)h^{(j)}(x).$$

Using the notation x^n to denote the falling factorial, it follows that

$$\begin{aligned} f^m(x) &= \sum_{j=0}^m \binom{m}{j} k^{m-j} (x-\alpha)^{k-m+j} (n-1-k)^j (x-\beta)^{n-1-k-j} \\ &= m! \frac{f(x)}{(x-\alpha)^m(x-\beta)^m} \sum_{j=0}^m \binom{k}{m-j} \binom{n-1-k}{j} (x-\alpha)^j (x-\beta)^{m-j}. \end{aligned}$$

Lemma 2: Let $0 \leq m \leq n-1$ be a fixed integer. Then

$$v_{n-1-m} = \sum_{j=0}^m (-1)^m \binom{k}{m-j} \binom{n-1-k}{j} \alpha^{m-j} \beta^j$$

and

$$(S(n)v)_{n-1-m} = \sum_{r=m}^{n-1} (-q)^m \binom{r}{m} p^{r-m} v_r.$$

Proof: The first result follows by computing the coefficient of x^{n-1-m} in $f(x)$ by multiplying $(x-\alpha)^k$ times $(x-\beta)^{n-1-k}$. The second result follows by computing the product of $S(n)$ and v .

Theorem: $S(n)v = \alpha^{n-1-k} \beta^k v$.

Proof:

$$\begin{aligned} (S(n)v)_{n-1-m} &= \sum_{r=m}^{n-1} (-q)^m \binom{r}{m} p^{r-m} v_r \\ &= \frac{(-q)^m}{m!} \sum_{r=m}^{n-1} v_r r^m p^{r-m} = \frac{(-q)^m}{m!} f^{(m)}(p) \\ &= \frac{(-1)^m (\alpha \cdot \beta)^m (p-\alpha)^k (p-\beta)^{n-1-k}}{m! (p-\alpha)^m (p-\beta)^m} \\ &= m! \cdot \sum_{j=0}^m \binom{k}{m-j} \binom{n-1-k}{j} (p-\alpha)^j (p-\beta)^{m-j} \\ &= \alpha^{n-1-k} \beta^k (-1)^m \sum_{j=0}^m \binom{k}{m-j} \binom{n-1-k}{j} \beta^j \alpha^{m-j} \\ &= \alpha^{n-1-k} \beta^k \sum_{j=0}^m (-1)^m \binom{k}{m-j} \binom{n-1-k}{j} \alpha^{m-j} \beta^j \\ &= \alpha^{n-1-k} \beta^k v_{n-1-m}. \end{aligned}$$

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AMS Classification Numbers: 11B65, 15A36, 15A18



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