

ON FIBONACCI SEQUENCES, GEOMETRY, AND THE m -SQUARE EQUATION

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1. INTRODUCTION

The general solution of Pythagoras' 3-square equation, namely $x^2 + y^2 = z^2$, for integer triples (x, y, z) , has been known since Euclid in the 4th century B.C. In 1988, Georg Schaaake and John Turner discovered a new way of finding all solutions, using the rational number tree to help find and classify them [5]. Their methods were heavily dependent on continued fractions, and hence closely related to the arithmetic of Fibonacci sequences.

Recently, when considering properties of certain triangles in \mathbf{R}^3 in relation to Fibonacci sequences, Turner discovered solutions for the 4-square equation $x^2 + y^2 + z^2 = w^2$ in integer quadruples (x, y, z, w) , with x, y, z, w each being a function of Fibonacci numbers. A simple number tree helped in this discovery too. The idea easily generalized (see Section 3, below), providing a two-variable identity which defined a more general class of integer solutions to the 4-square equation.

The two-variable identity that we found turned out to be a special case of one given by Catalan in 1877; it was discovered independently by Dainelli, and published also in 1877 (see [1], [2], [3]). Modern treatments of the 4-square equation do not refer to these identities, and the general solutions of it do not point to or suggest relationships with Fibonacci numbers (see, e.g., Sierpinski's book [6]).

We believe that our manner of finding such relationships may be new, and that the story of their discovery, from a triangle which we decided to call an *F-triangle*, will be found interesting. Moreover, we show how the vector geometric approach may be exploited, and discover further interesting results about sequences of F-triangles, presenting us with a variety of Fibonacci identities.

Further study of the 4-square identity led to discovery of infinite classes of integer-solutions, in terms of F_{n-1} , F_n , and F_{n+1} , for both the 3-square equation (Pythagoras') and the 5-square equation. Later we extended this process to provide solution-class formulas for the m -square equation, with $m = 6, 7, 8, \dots$. These formulas for the solutions of the infinite sequence of equations are presented in the final section of the paper.

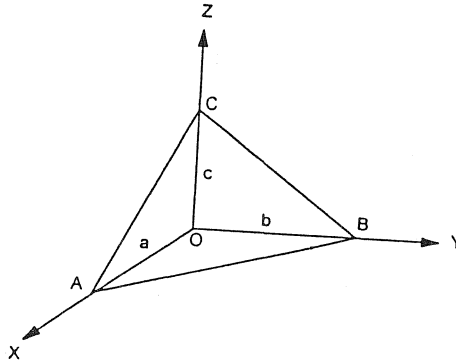
2. AREAS OF F-TRIANGLES AND 4-SQUARE SOLUTIONS

A triple of positive integers (a, b, c) will be associated with the triangle ABC , where $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$. The diagram below shows the arrangement of axes and

triangle. The area of the triangle, denoted by Δ , is easily computed from elementary vector analysis, thus:

$$\Delta = \frac{1}{2} |\underline{AB} \times \underline{AC}| = \frac{1}{2} |(-a, b, 0) \times (-a, 0, c)| = \frac{1}{2} |(bc, ac, ab)|.$$

Thus, $\Delta = \frac{1}{2} \sqrt{S}$, where $S = (ab)^2 + (bc)^2 + (ca)^2$. Any choice of a, b, c for which S is a square provides a solution of the 4-square equation, with $x = ab$, $y = bc$, and $z = ca$.



The case $a = F_{n-1}$, $b = F_n$, $c = F_{n+1}$, where F_n is the n^{th} Fibonacci number, is of special interest. It seems appropriate to call this an *F-Triangle*, where $n = 2, 3, \dots$. The area of the n^{th} F-triangle is then:

$$\Delta_n = \frac{1}{2} \sqrt{S_n}, \text{ with } S_n = (F_{n-1}F_n)^2 + (F_nF_{n+1})^2 + (F_{n-1}F_{n+1})^2.$$

As the following table indicates, S_n seems to be a perfect square.

n	2	3	4	5
S_n	3^2	7^2	19^2	49^2

The formula $S_n = (F_{2n-1} + F_{n-1}F_n)^2$ (or, equivalently, $S_n = (F_{n+1}^2 - F_nF_{n-1})^2$) was found by inspection. This corresponds to the class of solutions

$$x = F_{n-1}F_n, \quad y = F_nF_{n+1}, \quad z = F_{n-1}F_{n+1}, \quad w = F_{2n-1} + F_{n-1}F_n$$

of the 4-square equation $x^2 + y^2 + z^2 = w^2$. For $n = 2, 3, 4, 5$, the respective solutions (x, y, z, w) are $(1, 2, 2, 3)$, $(2, 3, 6, 7)$, $(6, 10, 15, 19)$, and $(15, 24, 40, 49)$, all of which happen to be primitive.

The formula given for S_n can be proved to be correct with algebraic manipulation and use of Fibonacci identities. This proof is omitted, however, since in the next section it will be discovered that the success of our approach owes to the fact that $c = a + b$, and not to the particular choice of a and b . This observation will lead to a simple proof of a useful theorem, and to additional classes of solutions of the 4-square equation.

3. MORE GENERAL CLASSES OF SOLUTIONS OF THE 4-SQUARE EQUATION

In Section 2, it was shown that the area Δ of the triangle ABC associated with the positive integer triple (a, b, c) was $\Delta = \frac{1}{2} \sqrt{S}$, where $S = (ab)^2 + (bc)^2 + (ca)^2$. If $c = a + b$, then:

$$\begin{aligned} S &= (ab)^2 + (b(a+b))^2 + ((a+b)a)^2 \\ &= a^4 + b^4 + 3a^2b^2 + 2(a^3b + b^3a)^2 \\ &= (a^2 + b^2 + ab)^2. \end{aligned}$$

This calculation gives us the following theorem.

Theorem: Let a and b be positive integers. Then

$$x = ab, \quad y = b(a+b), \quad z = a(a+b), \quad w = (a^2 + b^2 + ab)$$

is a solution of the 4-square equation $x^2 + y^2 + z^2 = w^2$. \square

Any general Fibonacci-type sequence $a, b, a+b, a+2b, 2a+3b, \dots$ thereby generates an infinite sequence of *general* F-triangles, and a corresponding infinite class of solutions of the 4-square equation.

Applications

(i) The F-triangles of Section 2 give $w = F_{n-1}^2 + F_n^2 + F_{n-1}F_n = F_{2n-1} + F_nF_{n-1}$, proving the result of Section 2.

(ii) The area of the triangle corresponding to the triple $(a, b, a+b)$ is $\Delta = (a^2 + b^2 + ab)/2$. Thus, the area is integral if and only if a and b are both even. Since no two consecutive Fibonacci numbers are even, the Fibonacci F-triangles of Section 2 all have half-integer areas; the first four of these are $3/2, 7/2, 19/2, 49/2$.

(iii) Another example, this time involving Lucas numbers, is the following. Using $(a, b, c) = (L_{n-1}, L_n, L_{n+1})$ gives:

$$(L_{n-1}L_n)^2 + (L_{n-1}L_{n+1})^2 + (L_nL_{n+1})^2 = (2L_{2n} + (-1)^{n-1})^2.$$

Hence, $x = L_{n-1}L_n, y = L_{n-1}L_{n+1}, z = L_nL_{n+1}$, and $w = 2L_{2n} + (-1)^{n-1}$ is another class of solutions to the 4-square equation.

(iv) Yet another example, this time involving both Fibonacci and Lucas numbers, is the following triangle, and identity. Using $(a, b, c) = (F_{n-1}, F_{n+1}, L_n)$ gives:

$$(F_{n-1}F_{n+1})^2 + (F_{n+1}L_n)^2 + (L_nF_{n-1})^2 = (F_{n-1}^2 + L_nF_{n+1})^2.$$

Hence, $x = F_{n-1}F_{n+1}, y = F_{n+1}L_n, z = L_nF_{n-1}$, and $w = F_{n-1}^2 + L_nF_{n+1}$ is another class of solutions to the 4-square equation.

4. SOME PROPERTIES OF F-TRIANGLES

There are several interesting geometric properties of the Fibonacci F-triangles defined in Section 2. Similar procedures for the more general F-triangles would lead to other Fibonacci and Lucas identities.

Area Differences: The area of the n^{th} Fibonacci F-triangle is $\Delta_n = (F_{2n-1} + F_{n-1}F_n)/2$.

Then the difference of successive triangle-areas is given by the remarkably simple formula

$$\Delta_{n+1} - \Delta_n = F_nF_{n+1}F_{2n+1}.$$

Area Ratio: The ratio of successive areas is:

$$\frac{\Delta_{n+1}}{\Delta_n} = \frac{F_n F_{n+1} + F_{2n+1}}{F_{n-1} F_n + F_{2n-1}}.$$

This ratio tends to α^2 as n tends to infinity, where α is the golden ratio.

An Application of Hero's Formula: Hero's well-known formula for the area of a triangle of sides u, v, w is $\sqrt{s(s-u)(s-v)(s-w)}$, where $s = (u+v+w)/2$. For the n^{th} Fibonacci triangle, the respective lengths are given (using the Pythagorean theorem and some Fibonacci number identities) by the formulas:

$$\begin{aligned} u = AB &= \sqrt{F_{n-1}^2 + F_n^2} = \sqrt{F_{2n-1}}, \\ v = BC &= \sqrt{F_n^2 + F_{n+1}^2} = \sqrt{F_{2n+1}}, \\ w = CA &= \sqrt{F_{n-1}^2 + F_{n+1}^2}. \end{aligned}$$

But we have already derived another formula for Δ_n , so equating the two expressions for this area gives us the rather amazing identity:

$$\frac{1}{2}(F_{n-1}F_n + F_{2n-1}) = \sqrt{s(s-\sqrt{F_{2n-1}})(s-\sqrt{F_{2n+1}})(s-\sqrt{F_{n-1}^2 + F_{n+1}^2})},$$

where $s = (u+v+w)/2$.

Applying the Law of Cosines: Let θ be the angle between AB and BC . Using the cosine formula $w^2 = v^2 + u^2 - 2vu \cos \theta$, and also direction ratios for AB and BC , we can find expressions for $\cos \theta$ in two ways. Equating these gives the following identity:

$$L_n^2 - L_{2n} = 2(F_{n-1}F_{n+1} - F_n^2).$$

5. CLASSES OF SOLUTIONS FOR THE GENERAL m -SQUARE EQUATIONS

Further study of the 4-square identity found as described in Section 2, together with application of formulas from [5, p. 97], led us to solutions, in terms of Fibonacci numbers, of both the ancient 3-square equation for sides of a right triangle (Pythagoras') and the 5-square equation. We then let m be the number of square terms in an equation, and sought general classes of solutions for $m \in \mathbb{N}$, $m > 2$.

Having obtained solutions for case $m = 5$ from those of the case $m = 4$, it quickly became obvious how we might extend the process indefinitely; that is, give solution classes for $m = 6, 7, 8$, and so on.

We should point out that the solution to the 3-square equation was obtained from the 4-square one by a backward deductive process, which is not worth spelling out here in detail. The solution's validity can be checked easily using elementary Fibonacci identities. Whereas the solutions for cases $m = 5, 6, 7, \dots$ follow in sequence, from the 4-square solution, using the method given below under solution (3).*

* If this method were to be applied to the solution of the 3-square equation, another, different infinite list of equation solutions would result for $m = 4, 5, 6, \dots$. We were concerned only to present one such list, and to concentrate on the $m = 4$ (geometric) case as pivotal.

We end the paper by showing these solution classes (or an algorithm to obtain them sequentially), for the sequence of m -square equations, all expressed in terms of three consecutive Fibonacci numbers.

It must be pointed out that none of these solution classes constitutes a general solution for its equation—each is merely an interesting infinite subclass of the whole set of solutions.

We invite the reader to share the authors' view that there is much poetry in the following list. Might we call the list *An Infinite Ode on Square Equations*, or perhaps, *A Square Dance in Fibonacci Numbers*?

Solutions to m -Square Equations

(1) **The equation:** $x^2 + y^2 = z^2$, $m = 3$.

Solution:

$$x = F_{n-1}^2 + F_n^2 + F_{n-1}F_n,$$

$$y = 2F_{n-1}F_n^2F_{n+1},$$

$$z = 2F_{n-1}F_n^2F_{n+1} + 1 = y + 1.$$

(2) **The equation:** $x^2 + y^2 + z^2 = w^2$, $m = 4$.

Solution:

$$x = F_{n-1}F_n,$$

$$y = F_{n-1}F_{n+1},$$

$$z = F_nF_{n+1},$$

$$w = F_{n-1}^2 + F_n^2 + F_{n-1}F_n.$$

(3) **The equation:** $x^2 + y^2 + z^2 + u^2 = v^2$, $m = 5$.

Solution:

$$x = F_{n-1}F_n,$$

$$y = F_{n-1}F_{n+1},$$

$$z = F_nF_{n+1},$$

$$u = \frac{1}{2}[(F_{n-1}^2 + F_n^2 + F_{n-1}F_n)^2 - 1],$$

$$v = \frac{1}{2}[(F_{n-1}^2 + F_n^2 + F_{n-1}F_n)^2 + 1].$$

We can easily extend these solution classes so that, for any given m , we can compute a solution class for the m -square equation in terms of three consecutive Fibonacci numbers. The method is as follows:

Suppose that we have a solution $(x_1, x_2, \dots, x_{m-2}, x_{m-1})$ for the $(m-1)$ -square equation. Then take the function $x_{m-1} = f(F_{n-1}, F_n, F_{n+1})$ from this solution, and form the identity:

$$x_{m-1}^2 \equiv \left[\frac{1}{2}(x_{m-1}^2 + 1) \right]^2 - \left[\frac{1}{2}(x_{m-1}^2 - 1) \right]^2.$$

Then, if $u = \frac{1}{2}(x_{m-1}^2 - 1)$ and $v = \frac{1}{2}(x_{m-1}^2 + 1)$, we can state that $(x_1, x_2, \dots, x_{m-2}, u, v)$ is an integer solution of the m -square equation, provided that u and v are integers, which they are iff x_{m-1} is odd.

Beginning with the proven case $m = 4$, applying this method supplies the solution to case $m = 5$ as shown above, since $w = F_{n-1}^2 + F_n^2 + F_{n-1}F_n$ in (2) is always odd. Then, applying the method again gives a solution for $m = 6$; and so on *ad infinitum*.

Note that each of the above solutions is expressed in terms of the three consecutive Fibonacci integers, F_{n-1} , F_n , and F_{n+1} . Of course, we could easily eliminate one of these from the formulas, since $F_{n-1} + F_n = F_{n+1}$; however, that would defeat one of the objectives of the paper—namely, to show how neat identities and solutions arise from consideration of three consecutive numbers taken from Fibonacci sequences.

As a final comment, we note that the method we have just given to extend our classes of solutions to give a general solution for the m -square case uses precisely the same identity which gives that solution to Pythagoras' equation (case $m = 3$), which is credited to Pythagoras himself. This same solution was also found by Fibonacci in Proposition 1 of his book *Liber Quadratorum* [Book of Squares] published in 1225 [4]. We are somewhat surprised that Fibonacci did not make our extension to the m -square equation in his book of squares. Perhaps he regarded it as trivially obvious! We wonder whether he would have enjoyed our connection of the solutions with Fibonacci numbers.

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REFERENCES

1. E. Catalan. *Nouv. Correspond. Math.* **3** (1877):29.
2. U. Dainelli. *Giornale di Mat.* **15** (1877):378.
3. L. E. Dickson. *History of the Theory of Numbers*. Vol. 2, p. 266. New York: Chelsea, 1952.
4. Leonardo Pisano. *Liber Quadratorum*, 1225. In English: *The Book of Squares*, trans. L. E. Sigler. New York: Academic Press, 1987.
5. A. G. Schaake & J. C. Turner. *A New Chapter for Pythagorean Triples*. New Zealand: Department of Mathematics, University of Waikato, 1989. 155 pages.
6. W. Sierpinski. *Elementary Theory of Numbers*, MM 42. Warsaw, 1964.

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