ALTERNATING SUMS OF FOURTH POWERS OF FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The Fibonacci and Lucas numbers are defined for all integers n as

$$\begin{cases} F_{n+1} = F_n + F_{n-1}, & F_1 = F_2 = 1, \\ L_{n+1} = L_n + L_{n-1}, & L_1 = 1,\, L_2 = 3. \end{cases}$$

Their Binet forms are $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, where α and β are the roots of $x^2 - x - 1 = 0$.

Inspired by the well-known sum

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \tag{1.1}$$

Clary and Hemenway [2] obtained factored closed-form expressions for all sums of the form $\sum_{k=1}^{n} F_{mk}^3$, where m is an integer. For example, they discovered

$$\sum_{k=1}^{n} F_{2k}^3 = \begin{cases} \frac{1}{4} F_n^2 L_{n+1}^2 F_{n-1} L_{n+2} & \text{if } n \text{ is even}, \\ \frac{1}{4} L_n^2 F_{n+1}^2 L_{n-1} F_{n+2} & \text{if } n \text{ is odd}, \end{cases} \tag{1.2}$$

and

$$\sum_{k=1}^{n} F_{4k}^3 = \frac{1}{8} F_{2n}^2 F_{2n+2}^2 (L_{4n+2} + 6). \tag{1.3}$$

Motivated by the results of Clary and Hemenway, we turned to fourth powers to see if similar factorizations could be obtained. In the case of nonalternating sums, we could find nothing to compare with the beautiful formulas of Clary and Hemenway. However, by experimenting with many numerical examples, we found the most interesting results when we considered alternating sums. We present these results in Section 3, and indicate our method of proof in Section 4. As noted in [2], once such identities are discovered, it is usually a comparatively routine matter to prove them. However, to assist us in the proofs, we have discovered a number of striking sums that involve the Lucas numbers, and we present these in Section 2.

2. PRELIMINARY RESULTS

We require the following results.

$$F_{n+k} + F_{n-k} = F_n L_k, \qquad k \text{ even}, \tag{2.1}$$

$$F_{n+k} + F_{n-k} = L_n F_k, \qquad k \text{ odd}, \tag{2.2}$$

$$F_{n+k} - F_{n-k} = F_n L_k, \qquad k \text{ odd}, \tag{2.3}$$

$$F_{n+k} - F_{n-k} = L_n F_k, \qquad k \text{ even}, \tag{2.4}$$
$$L_{n+k} + L_{n-k} = L_n L_k, \qquad k \text{ even}, \tag{2.5}$$
$$L_{n+k} + L_{n-k} = 5F_n F_k, \qquad k \text{ odd}, \tag{2.6}$$
$$L_{n+k} - L_{n-k} = L_n L_k, \qquad k \text{ odd}, \tag{2.7}$$
$$L_{n+k} - L_{n-k} = 5F_n F_k, \qquad k \text{ even}, \tag{2.8}$$
$$L_{2m} - 2 = L_m^2, \qquad m \text{ odd}, \tag{2.9}$$
$$L_{2m} + 2 = L_m^2, \qquad m \text{ even}, \tag{2.10}$$
$$L_{2m} + (-1)^{m+1}2 = 5F_m^2. \tag{2.11}$$

Identities (2.1)-(2.8) appear as (5)-(12) in Bergum and Hoggatt [1], while (2.9)-(2.11) can be proved with the aid of the Binet forms for F_n and L_n.

Throughout this paper $m \neq 0$ is an integer. To assist in our proofs, we also make use of four sums which involve Lucas numbers with even subscripts. If m is odd, we have

$$\sum_{k=1}^{n} L_{2mk} = \begin{cases} \dfrac{5F_{mn}F_{m(n+1)}}{L_m}, & n \text{ even}, \\[2ex] \dfrac{L_{mn}L_{m(n+1)}}{L_m}, & n \text{ odd}, \end{cases} \tag{2.12}$$

and

$$\sum_{k=0}^{n} L_{2mk} = \begin{cases} \dfrac{L_{mn}L_{m(n+1)}}{L_m}, & n \text{ even}, \\[2ex] \dfrac{5F_{mn}F_{m(n+1)}}{L_m}, & n \text{ odd}. \end{cases} \tag{2.13}$$

On the right sides of (2.12) and (2.13), the even and odd cases are reversed. Equally surprising, we have found that for m even

$$\sum_{k=1}^{n} (-1)^k L_{2mk} = \begin{cases} \dfrac{5F_{mn}F_{m(n+1)}}{L_m}, & n \text{ even}, \\[2ex] -\dfrac{L_{mn}L_{m(n+1)}}{L_m}, & n \text{ odd}, \end{cases} \tag{2.14}$$

and

$$\sum_{k=0}^{n} (-1)^k L_{2mk} = \begin{cases} \dfrac{L_{mn}L_{m(n+1)}}{L_m}, & n \text{ even}, \\[2ex] -\dfrac{5F_{mn}F_{m(n+1)}}{L_m}, & n \text{ odd}. \end{cases} \tag{2.15}$$

The proofs of (2.12)-(2.15) are similar. We illustrate the method by proving (2.13).

Proof of (2.13): Expressing L_{2mk} in Binet form and summing the resulting geometric progressions, we obtain

$$\begin{aligned}
\sum_{k=0}^{n} L_{2mk} &= \frac{\alpha^{2mn+2m}-1}{\alpha^{2m}-1} + \frac{\beta^{2mn+2m}-1}{\beta^{2m}-1} \\
&= \frac{L_{2mn+2m} - L_{2mn} + L_{2m} - 2}{L_{2m}-2} \\
&= \frac{L_{(2mn+m)+m} - L_{(2mn+m)-m} + L_m^2}{L_m^2} && \text{[by (2.9)]} \\
&= \frac{L_{2mn+m}L_m + L_m^2}{L_m^2} && \text{[by (2.7)]} \\
&= \frac{L_{(mn+m)+mn} + L_{(mn+m)-mn}}{L_m}.
\end{aligned}$$

Since m is odd, (2.13) follows from (2.5) and (2.6). □

3. THE MAIN RESULTS

We now present our main results. If m is even, then

$$\sum_{k=1}^{n} (-1)^k F_{mk}^4 = \frac{(-1)^n F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} - 4L_{2m}]}{5 L_m L_{2m}}, \tag{3.1}$$

$$\sum_{k=1}^{n} (-1)^k L_{mk}^4 = \frac{5 F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} + 4L_{2m}]}{L_m L_{2m}}, \quad n \text{ even}, \tag{3.2}$$

$$\sum_{k=0}^{n} (-1)^k L_{mk}^4 = -\frac{5 F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} + 4L_{2m}]}{L_m L_{2m}}, \quad n \text{ odd}. \tag{3.3}$$

We mention that (3.2) and (3.3) can be combined in a single sum as

$$\sum_{k=1}^{n} (-1)^k L_{mk}^4 = \frac{(-1)^n 5 F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} + 4L_{2m}]}{L_m L_{2m}} - 8(1 + (-1)^{n+1}).$$

On the other hand, if m is odd, then

$$\sum_{k=1}^{n} (-1)^k F_{mk}^4 = \frac{(-1)^n F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} + 4(-1)^{n+1} L_{2m}]}{5 L_m L_{2m}}, \tag{3.4}$$

$$\sum_{k=1}^{n} (-1)^k L_{mk}^4 = \frac{5 F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} + 4L_{2m}]}{L_m L_{2m}}, \quad n \text{ even}, \tag{3.5}$$

$$\sum_{k=0}^{n} (-1)^k L_{mk}^4 = -\frac{5 F_{mn} F_{m(n+1)} [L_m L_{mn} L_{m(n+1)} - 4L_{2m}]}{L_m L_{2m}}, \quad n \text{ odd}. \tag{3.6}$$

As before, (3.5) and (3.6) can be expressed as a single sum, but we choose to write them separately in order to present the right sides in factored form. This is the reason for the appearance of the zero lower limit.

4. THE METHOD OF PROOF

To illustrate the method, we prove (3.4). First, let n be even. In what follows, we note that since m is odd and $\alpha\beta=-1$, then $(\alpha\beta)^{mk}=(-1)^k$. Now

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k F_{mk}^4 &= \frac{1}{25}\sum_{k=1}^{n}(-1)^k(\alpha^{mk}-\beta^{mk})^4\\ &= \frac{1}{25}\sum_{k=1}^{n}(-1)^k(L_{4mk}-4(-1)^k L_{2mk}+6)\\ &= \frac{1}{25}\sum_{k=1}^{n}((-1)^k L_{4mk}-4L_{2mk}+6(-1)^k)\\ &= \frac{1}{25}\sum_{k=1}^{n}((-1)^k L_{4mk}-4L_{2mk}), \text{ since } n \text{ is even.}\end{aligned}$$

With the use of (2.12) and (2.14), this becomes

$$\begin{aligned}\frac{1}{25}\left[\frac{5F_{2mn}F_{2m(n+1)}}{L_{2m}}-\frac{20F_{mn}F_{m(n+1)}}{L_m}\right] &= \frac{1}{5}\left[\frac{F_{mn}L_{mn}F_{m(n+1)}L_{m(n+1)}}{L_{2m}}-\frac{4F_{mn}F_{m(n+1)}}{L_m}\right]\\ &= \frac{F_{mn}F_{m(n+1)}[L_mL_{mn}L_{m(n+1)}-4L_{2m}]}{5L_mL_{2m}}.\end{aligned}$$

If n is odd, then we have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k F_{mk}^4 &= \sum_{k=0}^{n}(-1)^k F_{mk}^4 \quad (\text{since } F_0=0)\\ &= \frac{1}{25}\sum_{k=0}^{n}((-1)^k L_{4mk}-4L_{2mk}+6(-1)^k)\\ &= \frac{1}{25}\sum_{k=0}^{n}((-1)^k L_{4mk}-4L_{2mk}), \text{ since } n \text{ is odd.}\end{aligned}$$

With the aid of (2.13) and (2.15), this sum becomes

$$\begin{aligned}\frac{1}{25}\left[\frac{-5F_{2mn}F_{2m(n+1)}}{L_{2m}}-\frac{20F_{mn}F_{m(n+1)}}{L_m}\right] &= -\frac{1}{5}\left[\frac{F_{mn}L_{mn}F_{m(n+1)}L_{m(n+1)}}{L_{2m}}+\frac{4F_{mn}F_{m(n+1)}}{L_m}\right]\\ &= -\frac{F_{mn}F_{m(n+1)}[L_mL_{mn}L_{m(n+1)}+4L_{2m}]}{5L_mL_{2m}},\end{aligned}$$

and this completes the proof. □

We remark that the proof of (3.1) is similar since the parities of n must be considered separately, but the proofs of the other results in Section 3 are more straightforward.

5. CONCLUSION

During the course of our investigation we discovered two further pairs of sums similar in character to (2.12)-(2.15) which we include here. If m is odd, then

$$\sum_{k=1}^{n}(-1)^k L_{2mk} = \frac{(-1)^n F_{mn} L_{m(n+1)}}{F_m}, \tag{5.1}$$

and

$$\sum_{k=0}^{n}(-1)^k L_{2mk} = \frac{(-1)^n L_{mn} F_{m(n+1)}}{F_m}. \tag{5.2}$$

If m is even, then

$$\sum_{k=1}^{n} L_{2mk} = \frac{F_{mn} L_{m(n+1)}}{F_m}, \tag{5.3}$$

and

$$\sum_{k=0}^{n} L_{2mk} = \frac{L_{mn} F_{m(n+1)}}{F_m}. \tag{5.4}$$

The Lucas counterpart of (1.1), which appears as I_4 in [3], is

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 = L_n L_{n+1} - L_0 L_1. \tag{5.5}$$

The right side of (5.5) suggests the notation $[L_j L_{j+1}]_0^n$.

We now make an observation about identity (3.4) and its Lucas counterpart. We have found that for $m = 1$ they can be expressed as

$$\sum_{k=1}^{n}(-1)^k F_k^4 = \frac{(-1)^n}{3} F_{n-2} F_n F_{n+1} F_{n+3}, \tag{5.6}$$

and

$$\sum_{k=1}^{n}(-1)^k L_k^4 = \left[\frac{(-1)^j}{3} L_{j-2} L_j L_{j+1} L_{j+3}\right]_0^n. \tag{5.7}$$

They can be proved quite effectively using the method outlined on page 135 of [2]. We illustrate by proving (5.7).

Let l_n denote the sum on the left side of (5.7), and let $r_n = \frac{(-1)^n}{3} L_{n-2} L_n L_{n+1} L_{n+3}$. Then

$$r_n - r_{n-1} = \frac{(-1)^n}{3} L_n (L_{n-2} L_{n+1} L_{n+3} + L_{n-3} L_{n-1} L_{n+2}). \tag{5.8}$$

Now, by using the recurrence satisfied by the Lucas numbers, we express L_{n-2}, L_{n+3}, L_{n-3}, L_{n-1}, and L_{n+2} in terms of L_n and L_{n+1}, and substitute in (5.8) to obtain

$$r_n - r_{n-1} = l_n - l_{n-1} = \frac{(-1)^n}{3} L_n^4.$$

Thus, $l_n - r_n = -r_0$, and this proves (5.7).

To conclude, we mention that for p real the sequences $\{U_n\}$ and $\{V_n\}$ defined for all integers n by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases}$$

generalize the Fibonacci and Lucas numbers, respectively. Identities (2.12)-(2.15), together with the results in Section 3, and (5.1)-(5.4) translate immediately to U_n and V_n. The reason is that if we replace F_n by U_n, L_n by V_n, and 5 by p^2+4, then U_n and V_n satisfy (2.1)-(2.11), upon which all our proofs are based. For example, if m is odd, (3.4) and (3.5) become, respectively,

$$\sum_{k=1}^{n}(-1)^k U_{mk}^4 = \frac{(-1)^n U_{mn}U_{m(n+1)}[V_m V_{mn} V_{m(n+1)} + 4(-1)^{n+1}V_{2m}]}{(p^2+4)V_m V_{2m}}, \tag{5.9}$$

and

$$\sum_{k=1}^{n}(-1)^k V_{mk}^4 = \frac{(p^2+4)U_{mn}U_{m(n+1)}[V_m V_{mn} V_{m(n+1)} + 4V_{2m}]}{V_m V_{2m}}, \quad n \text{ even.} \tag{5.10}$$

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