ON THE FIBONACCI NUMBERS AND THE DEDEKIND SUMS

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1. INTRODUCTION

As usual, the Fibonacci sequence $F=(F_n)$ is defined by $F_0=0$, $F_1=1$, and by the second-order linear recurrence sequence $F_{n+2}=F_{n+1}+F_n$ for $n\geq 0$. This sequence has many important properties, and it has been investigated by many authors. In this paper we shall attempt to study the distribution problem of Dedekind sums for Fibonacci numbers and obtain some interesting results. For convenience, we first introduce the definition of the Dedekind sum $S(h,q)$. For a positive integer q and an arbitrary integer h, we define

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer;}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various arithmetical properties of $S(h,k)$ can be found in [3], [4], and [6]. About Dedekind sums and uniform distribution, Myerson [5] and Zheng [7] have obtained some meaningful conclusions. However, it seems that no one has yet studied the mean value distribution of $S(F_n,F_{n+1})$, at least we have not found expressions such as $\sum S(F_n,F_{n+1})$ in the literature. The main purpose of this paper is to study the mean value distribution of $S(F_n,F_{n+1})$ and present a sharper asymptotic formula. That is, we shall prove the following main theorem.

Theorem: Let m be a positive integer, then we have

$$\sum_{n=1}^{m}S(F_n,F_{n+1})=-\frac{(\sqrt{5}-1)^2}{48}m+C(m)+O\left(\frac{1}{\alpha^{2m}}\right),$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, $C(m)$ is a constant depending only on the parity of m, i.e.,

$$C(m)=\begin{cases}\dfrac{1}{12}\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_{2n}F_{2n+1}}+\frac{1}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_n} & \text{if } m \text{ is an even number;}\\[2ex] \dfrac{1}{12}\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_{2n+1}F_{2n+2}}+\frac{1}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_n} & \text{if } m \text{ is an odd number.}\end{cases}$$

2. SOME LEMMAS

To complete the proof of the theorem, we need the following two lemmas.

Lemma 1: Let m be a positive integer, then we have

$$S(F_m,F_{m+1})+\frac{F_{m-1}}{12F_{m+1}}=\frac{1}{12}\left[\frac{1}{F_mF_{m+1}}-\frac{1}{F_{m-1}F_m}+\cdots+(-1)^{m-2}\frac{1}{F_2F_3}\right].$$

Proof: It is clear that $(F_m, F_{m+1}) = 1$, $m = 1, 2, 3, \ldots$, so, from the reciprocity formula of Dedekind sums (see [2] or [3]), we get

$$S(F_m, F_{m+1}) + S(F_{m+1}, F_m) = \frac{F_m^2 + F_{m+1}^2 + 1}{12F_mF_{m+1}} - \frac{1}{4}. \tag{1}$$

By the recursion relationship $F_{m+1} = F_m + F_{m-1}$ for $m > 0$, we have $S(F_{m+1}, F_m) = S(F_{m-1}, F_m)$. Thus,

$$\begin{aligned} S(F_m, F_{m+1}) + S(F_{m-1}, F_m) &= \frac{F_m^2 + F_{m+1}^2 + 1}{12F_mF_{m+1}} - \frac{1}{4} = \frac{1}{12}\left(\frac{F_m}{F_{m+1}} + \frac{F_{m+1}}{F_m} + \frac{1}{F_mF_{m+1}}\right) - \frac{1}{4} \\ &= \frac{1}{12}\left(\frac{F_{m-1}}{F_m} + \frac{F_m}{F_{m+1}} + \frac{1}{F_mF_{m+1}}\right) - \frac{1}{6} = \frac{1}{12F_mF_{m+1}} - \frac{F_{m-1}}{12F_{m+1}} - \frac{F_{m-2}}{12F_m}, \end{aligned}$$

so that

$$\begin{aligned} S(F_m, F_{m+1}) + \frac{F_{m-1}}{12F_{m+1}} &= \frac{1}{12F_mF_{m+1}} - \left[S(F_{m-1}, F_m) + \frac{F_{m-2}}{12F_m}\right] \\ &= \frac{1}{12F_mF_{m+1}} - \frac{1}{12F_{m-1}F_m} + \left[S(F_{m-2}, F_{m-1}) + \frac{F_{m-3}}{12F_{m-1}}\right] = \cdots \\ &= \frac{1}{12F_mF_{m+1}} - \frac{1}{12F_{m-1}F_m} + \frac{1}{12F_{m-2}F_{m-1}} - \cdots - (-1)^{m-2}\left[S(F_1, F_2) + \frac{F_0}{12F_2}\right]. \end{aligned}$$

It is clear that $S(F_1, F_2) = S(1, 1) = 0$ and $F_0 = 0$, so we obtain

$$S(F_m, F_{m+1}) + \frac{F_{m-1}}{12F_{m+1}} = \frac{1}{12F_mF_{m+1}} - \frac{1}{12F_{m-1}F_m} + \cdots + (-1)^{m-2}\frac{1}{12F_2F_3}.$$

This concludes the proof of Lemma 1.

Lemma 2: Let m be a positive integer, then we have

$$\sum_{n=1}^{m} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2}m + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Proof: From the second recursion relationship for F_n, we can easily deduce that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \quad \text{and} \quad \alpha F_m = F_{m+1} + \left(\frac{1}{\alpha}\right)^m.$$

From these identities, we get

$$\begin{aligned} \sum_{n=1}^{m} \frac{F_n}{F_{n+1}} &= \frac{1}{\alpha}\sum_{n=1}^{m} \frac{\alpha F_n}{F_{n+1}} = \frac{1}{\alpha}\sum_{n=1}^{m} \frac{F_{n+1} + \left(\frac{1}{\alpha}\right)^n}{F_{n+1}} \\ &= \frac{1}{\alpha}m + \sum_{n=1}^{m} \frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} = \frac{\sqrt{5}-1}{2}m + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right). \end{aligned}$$

This completes the proof of Lemma 2.

3. PROOF OF THE THEOREM

In this section we shall complete the proof of the theorem. First, let m be a positive integer, then from (1) we have

$$S(F_m, F_{m+1}) + S(F_{m+1}, F_m) = \frac{1}{12}\left(\frac{F_{m+1}}{F_m} + \frac{F_m}{F_{m+1}} + \frac{1}{F_m F_{m+1}}\right) - \frac{1}{4}$$

or

$$S(F_m, F_{m+1}) + S(F_{m-1}, F_m) = \frac{1}{12}\left(\frac{F_{m-1}}{F_m} + \frac{F_m}{F_{m+1}} + \frac{1}{F_m F_{m+1}}\right) - \frac{1}{6}$$

and

$$\sum_{n=1}^{m}[S(F_n, F_{n+1}) + S(F_{n-1}, F_n)] = \frac{1}{12}\sum_{n=1}^{m}\left[\frac{F_n}{F_{n+1}} + \frac{F_{n-1}}{F_n} + \frac{1}{F_n F_{n+1}}\right] - \frac{m}{6}.$$

Noting that

$$S(F_0, F_1) = S(0, 1) = 0 \quad \text{and} \quad F_0 = 0$$

so that

$$2\sum_{n=1}^{m} S(F_n, F_{n+1}) - S(F_m, F_{m+1}) = \frac{1}{12}\sum_{n=2}^{m}\frac{F_{n-1}}{F_n} + \frac{1}{12}\sum_{n=1}^{m}\frac{F_n}{F_{n+1}} + \frac{1}{12}\sum_{n=1}^{m}\frac{1}{F_n F_{n+1}} - \frac{m}{6};$$

hence,

$$2\sum_{n=1}^{m} S(F_n, F_{n+1}) = S(F_m, F_{m+1}) + \frac{1}{12}\frac{F_{m-1}}{F_{m+1}} + \frac{1}{6}\sum_{n=1}^{m}\frac{1}{F_{n+1}} + \frac{1}{12}\sum_{n=1}^{m}\frac{F_n}{F_n F_{n+1}} - \frac{2m+1}{12}. \tag{2}$$

Applying (2), Lemma 1, and Lemma 2, we obtain

$$\begin{aligned} 2\sum_{n=1}^{m} S(F_n, F_{n+1}) &= \frac{1}{12}\left[\frac{1}{F_m F_{m+1}} - \frac{1}{F_{m-1}F_m} + \cdots + (-1)^{m-2}\frac{1}{F_2 F_3}\right] \\ &\quad + \frac{1}{6}\left[\frac{\sqrt{5}-1}{2}m + \sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right)\right] + \frac{1}{12}\sum_{n=1}^{m}\frac{1}{F_n F_{n+1}} - \frac{2m+1}{12}. \end{aligned}$$

If m is an even number, then from the above we have

$$\begin{aligned} \sum_{n=1}^{m} S(F_n, F_{n+1}) &= \frac{\sqrt{5}-3}{24}m + \frac{1}{12}\sum_{n=1}^{m/2}\frac{1}{F_{2n}F_{2n+1}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right) \\ &= -\frac{(\sqrt{5}-1)^2}{48}m + \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{F_{2n}F_{2n+1}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right). \end{aligned}$$

If m is an odd number, then

$$\begin{aligned} \sum_{n=1}^{m} S(F_n, F_{n+1}) &= \frac{\sqrt{5}-3}{24}m + \frac{1}{12}\sum_{n=1}^{(m-1)/2}\frac{1}{F_{2n+1}F_{2n+2}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right) \\ &= -\frac{(\sqrt{5}-1)^2}{48}m + \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{F_{2n+1}F_{2n+2}} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\alpha}\right)^{n+1}}{F_{n+1}} + O\left(\frac{1}{\alpha^{2m}}\right). \end{aligned}$$

This completes the proof of the theorem.

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