ON THE EXTENDIBILITY OF THE SET {1, 2, 5}

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Let t be a nonzero integer and S a set of positive integers. We say that S is a P_t-set if, for any two distinct elements x and y of S, the integer $xy+t$ is a perfect square. A P_t-set is extendible if there exists a positive integer $a \notin S$ such that $S \cup \{a\}$ is still a P_t-set.

The problem of extending P_t-sets is very old and dates back to the time of Diophantus (see Dickson [5], p. 513). The most spectacular result in this area is due to Baker and Davenport [3] who showed that the P_1-set $\{1, 3, 8, 120\}$ is nonextendible. Since then, several authors have made efforts to give a characterization of the P_t-sets (see references).

The P_{-1}-set $\{1, 2, 5\}$ was studied by Brown [4] who proved that this set is nonextendible. His method is based on deep results of Baker [3] and techniques of Grinstead [10]. In this paper we give another proof of the nonextendibility of the P_{-1}-set $\{1, 2, 5\}$ using only elementary number theory.

Suppose that there exists an integer a such that $\{1, 2, 5, a\}$ is a P_{-1}-set. Then the following system of equations

$$\begin{cases} a-1=Y^2, \\ 2a-1=Z^2, \\ 5a-1=X^2, \end{cases} \tag{1}$$

has integral solutions X, Y, Z, in $\mathbf{Z}$. Without loss of generality, we can suppose X, Y, Z are in $\mathbb{N}^*$. Elimination of a in system (1) yields

$$\begin{cases} Z^2-2Y^2=1, \\ 2X^2-5Z^2=3. \end{cases} \tag{2}$$

Lemma 1: If system (1) admits a solution a, then there exists an integer k such that $a=12k+1$.

Proof: From system (1), it is clear that $a \equiv 1 \pmod 4$. The first equation in system (1) implies that $a \equiv \pm 1 \pmod 3$. If $a \equiv -1 \pmod 3$, then the second and third equations in system (1) imply that X and Z are both divisible by 3, which is impossible from the second equation in system (2). This gives $a \equiv 1 \pmod 3$. Then there exists an integer k such that $a=12k+1$. □

After replacing a by $12k+1$ in system (1), we obtain

$$\begin{cases} 12k=Y^2, \\ 24k+1=Z^2, \\ 60k+4=X^2. \end{cases} \tag{3}$$

System (3) yields

$$\begin{cases} 3k=y^2, \\ 24k+1=z^2, \\ 15k+1=x^2, \end{cases} \tag{4}$$

where $X = 2x$, $Y = 2y$, and $Z = z$. Therefore,

$$x^2 + 3y^2 = z^2, \text{ where } (x, y, z) = 1. \tag{5}$$

It is well known that the solutions of equation (5) are $x = \pm(n^2 - 3m^2), y = 2nm, z = n^2 + 3m^2$, with n and m two relatively prime integers.

The equation $y^2 = 3k$ implies $4n^2m^2 = 3k$ and $n^2 = \frac{3k}{4m^2}$. Therefore,

$$24k + 1 = z^2 = (n^2 + 3m^2)^2 = \left(\frac{3k}{4m^2} + 3m^2\right)^2$$

and

$$(24k + 1)16m^4 = 9k^2 + 144m^8 + 72m^4k.$$

Hence,

$$9k^2 - 312m^4k - 16m^4(1 - 9m^4) = 0. \tag{6}$$

Equation (6) is of the second degree in k with integer coefficients. Since k is an integer, the discriminant $12^2 13^2 m^8 + 144m^4(1 - 9m^4) = 144m^4(160m^4 + 1)$ of the left side in (6) should be the square of an integer. That is, $160m^4 + 1 = t^2$ for some $t \in \mathbb{N}$.

Lemma 2: The only solution of $160m^4 + 1 = t^2$ is $(m, t) = (0, \pm 1)$.

Proof: Clearly $m = 0$, $t = \pm 1$ is a solution for the equation $160m^4 + 1 = t^2$. Without loss of generality, we can suppose $m > 0$ and $t > 0$ [of course, if (m, t) is a solution, $(\pm m, \pm t)$ is also a solution for our equation]. Put $M = 2m$, then we obtain the equation

$$10M^4 + 1 = t^2, \ M > 0, \ t > 0. \tag{7}$$

From $(t - 1)(t + 1) = 10M^4$, we have either

$$\begin{cases} t - 1 = 2a^4, \ t + 1 = 80b^4, \ M = 2ab \\ \text{or} \\ t - 1 = 80b^4, \ t + 1 = 2a^4, \ M = 2ab \end{cases} \tag{8}$$

or

$$\begin{cases} t - 1 = 10a^4, \ t + 1 = 16b^4, \ M = 2ab \\ \text{or} \\ t - 1 = 16b^4, \ t + 1 = 10a^4, \ M = 2ab, \end{cases} \tag{9}$$

where a and b are two positive integers.

System (8) gives

$$a^4 - 40b^4 = \pm 1. \tag{10}$$

A congruence mod 4 shows that the minus sign on the left side of equation (10) can be rejected, and from $(a^2 - 1)(a^2 + 1) = 40b^4$, since $a^2 + 1$ and $a^2 - 1$ are not squares in $\mathbb{N}$ and $a^2 + 1$ is not divisible by 4, we have $a^2 + 1 = 2c^4$, $a^2 - 1 = 20d^4$, and $b = cd$, which gives

$$10d^4 + 1 = C^2, \text{ where } C = c^2. \tag{11}$$

Equation (11) is of the same type as equation (7), and since $d < a < M$, one can apply the method of descent.

System (9) gives

$$5a^4 - 8b^4 = \pm 1. \tag{12}$$

A congruence mod 8 shows that this is impossible. □

Theorem 1: The P_{-1}-set {1, 2, 5} is nonextendible.

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