THE FIRST 330 TERMS OF SEQUENCE A013583

Daniel C. Fielder
School of ECE, Georgia Tech, Atlanta, GA 30332-0250
dfielder@ee.gatech.edu

Marjorie R. Bicknell-Johnson
665 Fairlane Avenue, Santa Clara, CA 95051
marjohnson@earthlink.net
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1. INTRODUCTION

Let $R(N)$ be the number of representations of the natural number N as the sum of distinct Fibonacci numbers. The values of $R(N)$ are well recognized as the coefficient of x^N in the infinite product $\prod_{i=2}^{\infty}(1+x^{F_i})=(1+x)(1+x^2)(1+x^3)(1+x^5)\cdots=$

$$1+\mathbf{1x^1}+x^2+\mathbf{2x^3}+x^4+2x^5+2x^6+x^7+\mathbf{3x^8}+2x^9+2x^{10}+\cdots. \tag{1}$$

Combinatorially, each term $R(N)x^N$ counts the $R(N)$ partitions of N into distinct Fibonacci numbers. Some of the recursion properties of this sequence are investigated in [2]. The difficulties in producing this sequence are more computational than analytic in that usual generation methods quickly consume computer resources.

Our major interest is in the related sequence 1, 3, 8, 16, 24, 37, ..., $A_{n,\dots}$ whose n^{th} term is the least N such that $n=R(N)$, emphasized in boldface in (1) above. The general term of this sequence (see [8]), designated A013583 in Sloane's' on-line database of sequences, is still unknown. The 330 terms found in this note almost triple the 112 terms reported by Shallit [8]. Carlitz [3, 4], Klarner [7], and Hoggatt [4, 6], among others, have studied the representation of integers as sums of Fibonacci numbers and particularly Zeckendorf representations. The Zeckendorf representation of a natural number N uses only positive-subscripted, distinct, and non-consecutive Fibonacci numbers and is unique. We have used the Zeckendorf representation of N to write $R(N)$ in [1] and [2].

2. A PEEK AT A013583 FIRST

Let us begin by listing the terms of A013583 that we have computed. We will note very quickly why this sequence is so intractable. Table 1 lists 46 complete rows with 10 entries per row. The first 33 rows have no missing sequence terms; hence, 330 complete sequence terms. The first missing entry appears in the 34^{th} row as the yet unknown 331^{st} term. While there are necessarily missing terms in at least some of the remaining rows, there are also many useful calculated sequence terms. Our computer output concluded with a partial 47^{th} row with 5 unknown entries followed by the 446^{th} sequence term, 229971.

3. SOME OBSERVED AND COMPUTATIONAL PROPERTIES OF $\prod_{i=2}^{\infty}(1+x^{F_i})$

When $\prod_{i=2}^{\infty}(1+x^{F_i})$ is expanded, the terms are partitioned according to sets of palindromically arranged, successive $R(N)$ coefficients. For this reason, we refer to it as the *palindromic sequence*. The first few terms are given in (2) below.

TABLE 1. Terms of Sequence A013583 (Index 1 through 330 complete)

1	3	8	16	24	37	58	63	97	105
152	160	168	249	257	270	406	401	435	448
440	647	1011	673	723	715	1066	1058	1050	1092
1160	1147	1694	1155	1710	1702	2647	1846	1765	1854
2736	1867	2757	2744	2841	2990	2752	2854	2985	3019
4511	3032	6967	4456	3024	4477	4616	4451	7349	4629
7218	4917	4621	4854	4904	7179	7166	4896	7200	7247
7310	7213	7831	8187	7488	7205	11614	7480	7815	7857
7925	11593	18154	7912	11813	11682	11653	11619	7920	11669
11724	12669	12106	11661	12656	12093	18151	12648	18795	12792
19154	12101	20358	12711	12800	19099	20756	18761	18850	12813
18905	18871	46913	19557	19138	18858	19476	31134	20502	19565
20701	30579	18866	20832	21018	19578	47434	20463	20696	20777
20730	30414	30689	30359	30977	20743	47418	30503	47507	30702
30529	30969	30422	20735	30511	33176	30694	34684	47795	31676
53712	30524	49104	49201	33705	31689	47523	33108	49405	33286
49502	33574	49159	49112	31681	50091	49358	33278	33616	33561
50002	49489	53683	49366	49120	49222	49408	33553	49497	49434
49387	49667	53534	53670	53589	50107	54178	49400	50989	53615
54555	51222	56152	54521	51272	53581	124519	79607	49392	53856
79481	81141	79874	51264	79463	86241	53573	53848	54327	54225
124506	54293	81078	87927	80073	79476	80366	79853	82856	54280
80971	80086	131203	79942	82513	124433	124378	79913	124522	81073
79646	79879	54288	79984	79929	129221	82840	80361	129292	82882
125132	87694	82950	129538	86694	79921	86749	87131	131897	87681
129551	82937	128614	124417	130999	86686	128593	87673	142699	88016
129242	87817	128703	128831	129224	130004	82945	128606	129546	129402
129347	87126	87736	131177	87825	129216	130910	201246	133499	130012
142877	129326	128598	131190	134049	128873	129208	87838	129352	129250
201306	140539	129318	130025	140154	146927	140243	202466	142882	134185
131182	140298	142238	129305	140264	133494	142848	141861	216776	142780
140531	134104	141895	201314	134193	140251	142094	209286	208414	140958
217174	129313	211980	142225	142411	208244	209058	208037	209252	134206
______	212192	209668	209087	140259	140971	141856	142089	______	142170
208524	209396	142123	209634	209074	227408	______	209121	______	208079
212323	208511	______	209299	212268	142136	211985	209676	______	209409
227107	209210	217119	210396	227395	226777	______	227345	209236	227010
______	212260	______	208519	217436	209257	142128	209401	______	209218
______	231102	______	216881	210388	237867	217114	230958	______	217195
______	226968	217148	______	226929	______	______	209231	227112	228094
______	______	230416	______	230123	______	______	217161	______	______
______	______	______	226942	______	228107	______	230136	______	229704
______	______	______	229992	______	______	______	______	______	______
______	217153	______	______	______	______	______	______	______	228099
229696	230034	______	______	229979	______	______	______	______	______
______	______	______	______	______	______	______	______	______	______
______	______	______	______	______	229971				

$$\begin{aligned}&[1x^1]+x^2+[2x^3]+x^4+[2x^5+2x^6]+x^7+[3x^8+2x^9+2x^{10}+3x^{11}]+x^{12}\\&+[3x^{13}+3x^{14}+2x^{15}+4x^{16}+2x^{17}+3x^{18}+3x^{19}]+x^{20}+[4x^{21}+3x^{22}+3x^{23}\\&+5x^{24}+2x^{25}+4x^{26}+4x^{27}+2x^{28}+5x^{29}+3x^{30}+3x^{31}+4x^{32}]+x^{33}+[4x^{34}\\&+4x^{35}+3x^{36}+6x^{37}+3x^{38}+5x^{39}+5x^{40}+2x^{41}+6x^{42}+4x^{43}+4x^{44}+6x^{45}\\&+2x^{46}+5x^{47}+5x^{48}+3x^{49}+6x^{50}+3x^{51}+4x^{52}+4x^{53}]+x^{54}+[5x^{55}+\cdots\end{aligned} \tag{2}$$

Throughout this paper, we use the *floor* symbol $\lfloor x\rfloor$ to denote the **greatest** integer $\le x$ and the *ceiling* symbol $\lceil x\rceil$ to denote the **least** integer $\ge x$.

Square brackets identify coefficient palindromes. Palindromic sections share *external* boundaries of the form $1x^N$, $N=F_n-1$, consistent with $R(F_n-1)=1$ given in [3]. For data-handling and computation, we omitted the overlapping terms with unit coefficients and partitioned the expansion into palindromic sections which we call k-sections. The first term of a k-section is $\lfloor (k+2)/2\rfloor x^N$, $N=F_{k+2}$, and the last term is $\lfloor (k+2)/2\rfloor x^N$, $N=F_{k+3}-2$. In (2), observe coefficients [3 2 2 3] (for $k=4$) starting with x^8.

Since the second half of a k-section adds no new coefficients but merely repeats those of the first half in reverse order, we use $\frac{1}{2}k$-sections. If the number of terms is odd, we include the center term, which becomes the last term of the $\frac{1}{2}k$-section. The coefficient of the last term of the $\frac{1}{2}k$-section is always a power of 2.

The value of these central coefficients can be established using identity (3), which can be proved using mathematical induction.

$$\sum_{i=1}^{p} F_{3i+1}=F_{3p+1}+F_{3p-2}+\cdots+F_7+F_4=(F_{3p+3}-2)/2. \tag{3}$$

Thus,

$$R\left(\sum_{i=1}^{p} F_{3i+1}\right)=2^{p-1}R(F_4)=2^p=R((F_{3p+3}-2)/2) \tag{4}$$

by repeatedly applying $R(F_{n+3}+K)=2R(K)$, $F_n\le K<F_{n+1}$, and $R(F_4)=2$ from [2].

Take $k=3p-1$. The powers of x on the left and right internal boundaries of the k-section become F_{3p+1} and $F_{3p+2}-2$, and the central term has exponent $(F_{3p+1}+F_{3p+2}-2)/2=(F_{3p+3}-2)/2$, which is an integer since $2\mid F_{3p}$, and the coefficient is 2^p by (3) and (4).

Next, take $k=3p$. The central pair of terms have exponents of x given by $(F_{3p+4}-2-1)/2$ and $(F_{3p+4}-2+1)/2$, which are integers since F_{3p+4} is odd. We can establish the values of A and B below by mathematical induction:

$$F_{3p+2}+F_{3p-1}+\cdots+F_8+F_5=(F_{3p+4}-3)/2=A, \tag{5}$$

$$F_{3p+2}+F_{3p-1}+\cdots+F_8+F_5+F_2=(F_{3p+4}-1)/2=B. \tag{6}$$

By again applying $R(F_{n+3}+K)=2R(K)$ and $R(F_4)=2$ to (5) and (6),

$$R(A)=2^{p-1}R(F_5)=2^{p-1}(2)=2^p, \tag{7}$$

$$R(B)=2^pR(F_2)=2^p(1)=2^p, \tag{8}$$

so that $R(A)=R(B)=2^p$.

In the same way, when $k=3p+1$, the two central terms have equal coefficients given by 2^p. This establishes $2^{\lfloor\frac{k+1}{3}\rfloor}$ as the coefficient of the right boundary of a $\frac{1}{2}k$-section for all k.

Also from [2], we can apply $R(F_n) = [n/2] = R(F_{n+1} - 2)$ to the first and last terms of the bracketed palindromic sequences, and $R(F_n - 1) = 1$ explains the overlapping external boundaries of the k-sections.

The only practical way available at present to find the j^{th} term of A013583 is to search for the *first* appearance of j as a coefficient in the palindromic sequence and to record the corresponding exponent of x as the j^{th} term in A013583. Table 2 lists numerical properties of k-sections useful for setting up and checking our computational procedures.

TABLE 2. Numerical Parameters of Palindrome Sequence ($1 \le k \le 26$)

1	2	3	4	5	6	7	8	9	10	11
1		1	2	1	2	1	2	2	0	0
2		2	3	2	3	2	3	4	1	1*
3	4	2	5	2	5	2	6	7	2	1
4	7	3	8	2	9	3	11	12	4	2
5	12	3	13	4	16	3	19	20	7	4*
6	20	4	21	4	26	4	32	33	12	6
7	33	4	34	4	43	4	53	54	20	10
8	54	5	55	8	71	5	87	88	33	17*
9	88	5	89	8	115	5	142	143	54	27
10	143	6	144	8	187	6	231	232	88	44
11	232	6	233	16	304	6	375	376	143	72*
12	376	7	377	16	492	7	608	609	232	166
13	609	7	610	16	797	7	985	986	376	188
14	986	8	987	32	1291	8	1595	1596	609	305*
15	1596	8	1597	32	2089	8	2582	2583	986	493
16	2583	9	2584	32	3381	9	4179	4180	1596	798
17	4180	9	4181	64	5472	9	6763	6764	2583	1292*
18	6764	10	6765	64	8854	10	10944	10945	4180	2090
19	10945	10	10946	64	14327	10	17709	17710	6764	3382
20	17710	11	17711	128	23183	11	28655	28656	10945	5973*
21	28656	11	28657	128	37511	11	46366	46367	17710	8855
22	46367	12	46368	128	60695	12	75023	75024	28656	14328
23	75024	12	75025	256	98208	12	121391	121392	46367	23184*
24	121392	13	121393	256	158904	13	196416	196417	75024	37512
25	196417	13	196418	256	257113	13	317809	317810	121392	60696
26	317810	14	317811	512	416019	14	514227	514228	196417	98209

1	Value of k.	k
2	Power of x of left <u>external</u> boundary of k- or $\frac{1}{2}k$-sections.	$F_{k+2} - 1$
3	Integer coefficient of left <u>interior</u> boundary of k- or $\frac{1}{2}k$-sections.	$\lfloor \frac{k+2}{2} \rfloor$
4	Power of x of left interior boundary of k- or $\frac{1}{2}k$-sections.	F_{k+2}
5	Integer coefficient of right boundary of $\frac{1}{2}k$-section.	$2^{\lfloor \frac{k+1}{3} \rfloor}$
6	Power of x of right interior boundary of $\frac{1}{2}k$-section.	$\lfloor \frac{F_{k+4}-2}{2} \rfloor$
7	Integer coefficient of right interior boundary of k-section.	$\lfloor \frac{k+2}{2} \rfloor$
8	Power of x of right interior boundary of k-section.	$F_{k+3} - 2$
9	Power of x of right exterior boundary of k-section.	$F_{k+3} - 1$
10	Number of terms in k-section.	$F_{k+1} - 1$
11	Number of terms in $\frac{1}{2}k$-section. When 10 is odd, * indicates $\frac{1}{2}k$-section ends with unique center term of the k-section.	$\lceil \frac{F_{k+1}-1}{2} \rceil$

4. SUCCESSIVELY BETTER WAYS OF GETTING DATA FROM $\prod_{i=2}^{\infty}(1+x^{F_i})$

For small k-sections we inspected each successive printout by hand to select the first occurrence of each coefficient value. We found the first 112 terms in computing for $k \le 18$.

For further reduction in data handling we described the entries of $\frac{1}{2}k$-sections as {coefficient, power of x} pairs. Since only the unique {coefficient, smallest power of x for that coefficient} pairs from each $\frac{1}{2}k$-section qualify as potential pairs for A013583, we eliminated all pairs with duplicate "coefficient" portions except that pair with the least power of x. At the same time, the surviving pairs per $\frac{1}{2}k$-section emerge sorted by increasing coefficient size.

As an example, each line of (9) contains $\frac{1}{2}k$-section data, reduced and sorted as suggested. By suppressing the pairs that do not qualify, the highlighted {coefficient, powers of x} pairs for A013583 are immediately evident.

$$
\begin{aligned}
&\{\mathbf{\{1,1\}}\},\ k=1\\
&\{\mathbf{\{2,3\}}\},\ k=2\\
&\{\{2,5\}\},\ k=3\\
&\{\{2,9\},\ \mathbf{\{3,8\}}\},\ k=4\\
&\{\{2,15\},\{3,13\},\ \mathbf{\{4,16\}}\},\ k=5\\
&\{\{2,25\},\{3,22\},\{4,21\},\ \mathbf{\{5,24\}}\},\ k=6\\
&\{\{2,41\},\{3,36\},\{4,34\},\{5,39\},\ \mathbf{\{6,37\}}\},\ k=7\\
&\{\{2,67\},\{3,59\},\{4,56\},\{5,55\},\{6,60\},\ \mathbf{\{7,58\},\{8,63\}}\},\ k=8\\
&\{\{2,109\},\{3,96\},\{4,91\},\{5,89\},\{6,98\},\{7,94\},\{8,92\},\ \mathbf{\{9,97\},\{10,105\}}\},\ k=9
\end{aligned}
\tag{9}
$$

However, we were at the memory limit of our personal computer. We had to find new Fibonacci approaches to continue. When we found a way to let the *indices* of the Fibonacci numbers guide the computations in place of the Fibonacci numbers themselves, we had a fresh start with tremendously reduced computational requirements. The interaction between the Fibonacci numbers and their integer indices here is not the same as the divisibility properties noted in the many past studies of Fibonacci entry points and their periods. We needed formulas developed in [2] relating $R(N)$ to the Zeckendorf representation of N. By looking deeper into the structure of Fibonacci indices, we removed a core of redundancy to speed up and shorten our calculations and developed an improved way of assembling data and discarding duplicate data. We proceeded to calculate the remainder of the 330 terms of A013583 that you see in this paper. Even with our best available computation techniques, described below, we found size and time requirements to be impracticable for calculations beyond $k = 25$.

5. EXPLORING NEW WAYS TO FIND COEFFICIENTS OF k-SECTIONS

Since the combinatorial interpretation of the coefficients of the palindromic sequence is the number of partitions of the power of x into distinct Fibonacci members, we explore that point of view. We will use results of selected numerical examples to imply a general case. In the partial expansion of $\prod_{i=2}^{\infty}(1+x^{F_i})$ in (2), we observe the term $4x^{43}$, which tells us there are 4 partitions of 43 into distinct Fibonacci numbers. As is well known, 43 has the unique Zeckendorf representation, $43 = F_9 + F_6 + F_2 = 34+8+1$, where we rule out adjacent Fibonacci indices. As additional visual information, $F_9 = 34$ is the power of x of the left boundary of the k-section to which 43 belongs, and $F_{k+2} = F_9 = 34$ is the only Fibonacci power of x in its k-section, thus, $k = 7$. In general, we can represent any power in a k-section by its Zeckendorf representation which starts with F_{k+2}.

In [5], Fielder developed new *Mathematica*-oriented algorithms and programs for calculating and tabulating Zeckendorf representations and calculated the first 12,000 representations. We imbedded the algorithms in our work where needed. Reference [5] and *Mathematica* programs are available from Daniel C. Fielder. The indices in the Zeckendorf representation of an integer N give formulas for finding $R(N)$, as reported in [2]. We next describe how the indices are applied to our computer programs.

We noted earlier that the power of x of the first term of a k-section is not only a predictable Fibonacci number, but is the only power of x in that k-section which is a Fibonacci number. Because of the Fibonacci recursion, $F_{n+2} = F_{n+1} + F_n$, it is very easy to partition any Fibonacci number into distinct Fibonacci members. As an example, we represent the partition of $F_9 = 34$ as successive triangular arrays in (10):

$$\begin{array}{cccc} & & & 34 \\ & & 13 & 21 \\ & 5 & 8 & 21 \\ 2 & 3 & 8 & 21 \end{array} \qquad \begin{array}{cccc} & & & F_9 \\ & & F_7 & F_8 \\ & F_5 & F_6 & F_8 \\ F_3 & F_4 & F_6 & F_8 \end{array} \qquad \begin{array}{cccc} & & & 9 \\ & & 7 & 8 \\ & 5 & 6 & 8 \\ 3 & 4 & 6 & 8 \end{array} \tag{10}$$

The first array consists of the partition integers, the second consists of the Fibonacci number symbols with subscripts, and the third consists of Fibonacci *indices* only.

The enumeration of sequence subscripts for powers in general involves interaction among the restricted partitions of the several Fibonacci numbers used in the Zeckendorf representation. Computations controlled by the indices of the right array have advantages of symmetry. For example, the left-descending diagonal will always consist of all the consecutive odd or even integers starting with the index of the Fibonacci number to be partitioned. (Recall that we do not admit a 1 index.) Once the diagonal of odd (or even) indices is in place, the remaining column lower entries are all one less than their diagonal entry. The number of restricted partitions is the floor of half the largest index. In the example, $\lfloor \frac{9}{2} \rfloor = 4$ partitions. If the power of x were the single F_{k+2}, the number of partitions and, thereby, the coefficient would be $\lfloor \frac{k+2}{2} \rfloor$.

When we consider our example $43 = F_9 + F_6 + F_2$, we represent the individual partitions as three triangles of Fibonacci indices with the Zeckendorf Fibonacci indices as apexes. (The order from low to high is a computational preference.)

$$\begin{array}{c} 2 \end{array} \qquad \begin{array}{ccc} & & 6 \\ & 4 & 5 \\ 2 & 3 & 5 \end{array} \qquad \begin{array}{cccc} & & & 9 \\ & & 7 & 8 \\ & 5 & 6 & 8 \\ 3 & 4 & 6 & 8 \end{array} \tag{11}$$

By distributing each set of rows over the others, 12 sets of indices are found as the *Mathematica* string:

$$\begin{array}{l} \{9,6,2\}, \{9,4,5,2\}, \{9,2,3,5,2\}, \{7,8,6,2\}, \{7,8,4,5,2\}, \{7,8,2,3,5,2\}, \\ \{5,6,8,6,2\}, \{5,6,8,4,5,2\}, \{5,6,8,2,3,5,2\}, \{3,4,6,8,6,2\}, \\ \{3,4,6,8,4,5,2\}, \{3,4,6,8,2,3,5,2\}. \end{array} \tag{12}$$

Each set of indices of (12) evokes a partition of 43 into Fibonacci numbers having those indices. There are $1 \times 3 \times 4 = 12$ such partitions. Thus, we can use Zeckendorf representations both to count and to name partitions consisting solely of nonzero Fibonacci numbers. The results in (12) also suggest a very simple way to find the coefficients of the expansion $\prod_{i=2}^{\infty}[1/(1-x^{F_i})]$.

Our first computational improvement over direct expansion of (1) is given by our *Mathematica* program 10229601.ma. This program accepts 43, the power of x, and returns the coefficient 4 by using the equivalent of (11) to find (12) and then discarding sets with duplicate indices. The 4 sets of indices counted by 10229601.ma in the example are:

$$\{2,6,9\}, \{2,4,5,9\}, \{2,6,7,8\}, \{2,4,5,7,8\} \tag{13}$$

By using 10229601.ma in a loop, selected ranges of powers can be probed for power-coefficient pairs.

As the size of the powers increased, however, even 10229601.ma could not match the demands on it. This is because the distribution of indices in 10229601.ma takes place over all triangles, and memory is not released to be used again until the *end* of the computations. As an improvement, we distributed the index integers over the first two triangles on the right and eliminated sets with repeated integers. We applied this result to the next triangle alone, make the reductions, and repeated the process over the remaining triangles one at a time. The memory and time savings were substantial. In spite of the new computational advantages, the distribution was still over all of each triangle. With full triangle distribution, however, it is possible that there may be partitions with arbitrary length runs of repeating index integers. Since we want to count partitions with *no* repeating members, producing partitions through full distribution is not an optimum strategy.

Our next improvement restricted repeating members to a fixed and predictable limit per partition. We retained our earlier size order of the index triangles and eliminated enough lower rows so that the smallest member of a higher-order triangle is either equal to or just greater than the largest (or apex) member of its immediate lower-order neighbor. For illustration, we show the set of *partial* index triangles obtained from suitable modification of (11):

$$\begin{array}{ccccccc} 2 & & & 6 & & & 9 \\ & & 4 & 5 & & 7 & 8 \\ & 2 & 3 & 5 & & & \end{array} \tag{14}$$

Now, when distribution is made over all partial triangles, triple or higher repeats of individual integers cannot occur. The only possibility of a repeated integer lies between the least integer of a triangle and the greatest integer of its immediate left neighbor. This means that when repeats occur, there is at most one pair of those integers per partition. In fact, if each Zeckendorf representation index differed from the preceding index by an *odd* integer, there would be **no** repeating partition members, and the distribution operation on the partial triangles would immediately yield the integer indices of the desired set of Fibonacci partitions.

As proved in [2], $R(N)$ can be written immediately by repeatedly applying the formulas:

$$R(F_{n+2k+1}+K) = (k+1)R(K), \ F_n \le K < F_{n+1}, \tag{15}$$

$$R(F_{n+2k}+K) = kR(K) + R(F_{n+1}-K-2). \tag{16}$$

In our example, the distribution and reduction process yield integer sets $\{2,6\}$, $\{2,4,5\}$ from the first two reduced triangles. The process continues to the third partial triangle and produces the final $\{2,6,9\}, \{2,4,5,9\}, \{2,6,7,8\}, \{2,4,5,7,8\}$. Our program 10229601.ma incorporates the concept of partial triangles along with previous improvements. It was the first sufficiently robust program for handling k values of 24 and especially 25, necessary to obtain coefficients from the palindromic sequence to complete the last of the 330 terms of A013583. Next we study the 330 terms from Table 1.

6. THE 330 PAIRS $\{n, A_n\}$ SORTED BY INTERVALS

Returning to Table 1 which lists $\{n, A_n\}$ and also gives all known values of $A_n < F_{28}$, we sort the data by intervals as given in our k-sections. In Table 3 we select all $\{n, A_n\}$ such that $F_m \leq A_n < F_{m+1}$ and sort by increasing index values. For consistency with the terminology of [1] and [2], we take $m = k+2$.

TABLE 3. Indices n for $\{n, A_n\}$ Sorted by Intervals
$F_m \leq A_n < F_{m+1}$, $16 \leq m \leq 27$

1	2	3	4	5	6	7	Missing values for n (partial list)
16	987	8	23	32	34	1	33
17	1597	7	33	36	42	2	37, 41
18	2584	12	37	50	55	3	51, 53, 54
19	4181	11	51	52	68	5	53, 59, 61, 66, 67
20	6765	19	53	76	89	7	77, 82, 83, 85-88
21	10946	19	77	82	110	9	83, 97, 99, 101, 103, 106-109
22	17711	28	83	112	144	15	113, 118, 122, 127, 132-135, 137-143
23	28657	27	118	112	178	22	113, 127, 137, 139, 149, 151, 153, 154, 157, 159, 161, 163-164, 166-167, 171-177
24	46368	50	113	196	233	27	197, 198, 201-203, 205, 206, 211, 213-219 221-232
25	75025	43	198	196	288	39	197, 211, 223, 226, 227, 229, 236, 239, 241, 244, 249, 251, 253-255, 257, 259, 261, 263-266, 268-271, 274, 276-287
26	121393	76	197	277	377	52	278, 291, 298, 309, 314, 318, 319, 321, 323, 326-329, 331-334, 339, 341, 342, 344-355, 357-376
27	196418	72	278	330	466	69	331, 339, 347, 349, 353, 359, 367, 371, 373, 379, 381, 383, 389, 391, 394, 396, 397, 401, 402, 404, 406, 407, 409-413, 415, 417, 419, 421-423, 425-431, 433-439, 443, 444, 446-465

Column descriptions:

1	Value of m which defines the interval.
2	F_m
3	Number of pairs of $\{n, A_n\}$ in interval.
4	Smallest index n appearing in interval.
5	Every index n less than or equal to this number has appeared by interval's end.
6	Largest index n appearing in interval.
7	Number of missing indices less than the largest n in the interval.

Notice that the largest index n in each interval is a Fibonacci number or twice a Fibonacci number. If $m=2p$, the largest index is $n=F_{p+1}$; if $m=2p+1$, $n=2F_p$.

In every k-section that we computed beyond $k=12$, some indices were missing and appear for the first time in a later k-section. However, the "missing values" appear in an orderly way. The indices $n=F_{p+1}-1$ and $n=2F_p-1$ always are missing values for the respective intervals $m=2p$ and $m=2p-1$. (We note in passing that $n=112$ was the highest index available before the disclosures of this paper, and $m=20$ is complete for n through 112.)

Putting this all together, the first appearance of $n=F_{p+1}$ is for $A_n=F_{p+1}^2-1$ in the interval $F_{2p}<A_n<F_{2p+1}$, and the list of indices is complete for $n \le F_p$. The first appearance of $n=2F_p$ is for $A_n=F_{p+3}F_p-1+(-1)^{p+3}$ in the interval $F_{2p+1} \le A_n < F_{2p+2}$, and the indices are complete for $n \le 2F_{p-2}$. The first appearances of F_k and $2F_k$ are discussed in [1].

We notice that, if n is the largest index which appears for A_n in the interval $F_m \le A_n < F_{m+1}$, then the indices $n-1$, $n-2$, $n-3, \ldots, n-(F_{\lceil \frac{m}{2} \rceil -5}-1)$ are missing values.

The values for A_n are not a strictly increasing sequence if sorted by index, as can be seen from Table 1. However, if $F_p < n \le F_{p+1}$, then $F_{2p-1} < A_n < F_{2p+4}$. If n is prime, then $F_{2p} < A_n < F_{2p+1}$ or $F_{2p+2} < A_n < F_{2p+3}$. In fact, if n is prime, the Fibonacci numbers used in the Zeckendorf representation of A_n are all even subscripted.

We found palindromic subsequences and fractal-like recursions in tables of $\{n, A_n\}$. We developed many formulas relating $R(N)$ and the Zeckendorf representation of N, but we still cannot describe a general term for $\{n, A_n\}$. The formulas we developed and the programming data we generated each extended our knowledge while suggesting new approaches. Theory and application worked hand-in-glove throughout this entire project.

7. POSTSCRIPT AND AFTERMATH

After all the 330 consecutive terms and many other nonconsecutive terms of A013583 were calculated and recorded, and much of the paper completed, we stumbled onto a very simple *Mathematica*-implemented algorithm which uses the combinatorial principle of Inclusion-Exclusion to find the coefficients of $\prod_{i=2}^{\infty}(1+x^{F_i})$ for powers of x. While too late to help us gather data for the 330 terms, it provides a reassuring check on the work already completed, and should prove an invaluable aid in our continuing assault on sequence A013583. The *Mathematica* algorithm implementation is many times faster than that used for getting the 330 terms of A013583. Would you believe that a preliminary trial program with the new algorithm verified the coefficient of

$$x^{961531714240472069833557386959154606040263}$$

as 1475739525896764129280 in 2.62 seconds on a 133-Mhz PowerMac 7200 running *Mathematica*, version 2.2? Table 2 verifies this result because the power of x is that of column $\boxed{6}$ for $k=200$, while the coefficient matches the known value in column $\boxed{5}$ for $k=200$ in Table 2. Our paper describing the algorithm and its application has been reviewed and accepted for presentation at, and inclusion in, the proceedings of SOCO'99, Genoa, Italy, June 1-4, 1999.

A short paper outlining the Fibonacci and Zeckendorf algorithms of [5] has been accepted for presentation at the Southeastern MAA annual regional meeting in Memphis, TN, March 12-13, 1999.

We are also very optimistic about the ongoing development of an algorithm, hopefully with *Mathematica* implementation, which will generate terms of A013583 directly. Preliminary results have been most encouraging. The algorithm is based on ideas gathered from this note and reference [2].

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