CONGRUENCES RELATING RATIONAL VALUES OF BERNOULLI AND EULER POLYNOMIALS

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1. INTRODUCTION

For $n \in \mathbf{N}$, where $\mathbf{N} = \{0, 1, 2, \dots\}$, the Bernoulli polynomials, $B_n(t)$, are defined by means of the generating function

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1}$$

Some of the more important properties of these polynomials include

$$B_n(t+1) - B_n(t) = nt^{n-1}, \tag{2}$$

$$B_n(1-t) = (-1)^n B_n(t), \tag{3}$$

each of which follows from (1). From (2) we can derive

$$B_n(m) - B_n(0) = n\sum_{j=0}^{m-1} j^{n-1},$$

holding for all positive integers m. We define the Bernoulli numbers, B_n, by $B_n = B_n(0)$, from which (1) allows us to write

$$B_n(t) = \sum_{m=0}^{n}\binom{n}{m}B_{n-m}t^m.$$

Note that we obtain the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, ..., and $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$, we have

$$B_n = -\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m}B_m.$$

Perhaps the most fundamental property of Bernoulli numbers is the von Staudt-Clausen theorem which states that, for even positive n, the quantity

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p}$$

is an integer. This implies that, for such n, the denominator of B_n is square-free.

The Euler polynomials, $E_n(t)$, $n \in \mathbf{N}$, are defined by means of the generating function

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(t)\frac{x^n}{n!}, \quad |x| < \pi. \tag{4}$$

Each can be expanded in terms of Bernoulli numbers according to

$$E_n(t) = \sum_{m=0}^{n} \binom{n}{m} 2(1-2^{m+1}) \frac{B_{m+1}}{m+1} t^{n-m}.$$

Euler polynomials satisfy the identities

$$E_n(t+1) + E_n(t) = 2t^n, \tag{5}$$

$$E_n(1-t) = (-1)^n E_n(t), \tag{6}$$

each following from (4). From (5) we see that, for positive integers m,

$$E_n(m) - (-1)^m E_n(0) = 2\sum_{j=0}^{m-1} (-1)^{m-1-j} j^n.$$

The Euler numbers, E_n, are defined by $E_n = 2^n E_n(1/2)$. Each $E_n \in \mathbb{Z}$ (see [10], p. 53), and as a result of (6) we must have $E_n = 0$ whenever n is odd.

There are three particular identities, known as multiplication identities, associated with the Bernoulli and Euler polynomials. They enable one to rewrite a particular value of one of these polynomials in terms of a sum of a variety of values of either the same or another such polynomial. We present them as follows with the assumption that, for each, q is a positive integer. For the Bernoulli polynomials, we have Raabe's identity [12],

$$B_n(qt) = q^{n-1} \sum_{j=0}^{q-1} B_n\left(t + \frac{j}{q}\right), \tag{7}$$

which follows from (1). For q odd, the Euler polynomials satisfy

$$E_n(qt) = q^n \sum_{j=0}^{q-1} (-1)^j E_n\left(t + \frac{j}{q}\right), \tag{8}$$

which follows from (4). Finally, for q even,

$$E_{n-1}(qt) = -\frac{2q^{n-1}}{n} \sum_{j=0}^{q-1} (-1)^j B_n\left(t + \frac{j}{q}\right), \tag{9}$$

which follows from (1) and (4).

The problem of studying Bernoulli and Euler polynomials at values in $\mathbb{R}$ is tantamount to that of considering the polynomials in certain intervals of $\mathbb{R}$. From (2) and (5) we see that we can reduce this problem to that of considering the polynomials in $[0, 1)$. Utilizing (3) and (6) allows us further to reduce this to the interval $[0, 1/2]$. Because of this it becomes a point of interest to consider the polynomials at various "special" values of t in $[0, 1/2]$, especially at rational t. Applications of (7), (8), and (9) enable us to find relations between values of these polynomials at several rationals within this interval.

Let us now consider the known values of these polynomials. As we have seen, $B_n(0) = B_n$ and $E_n(1/2) = 2^{-n} E_n$ for each $n \geq 0$. The following can be derived from (7)-(9) for all $n \geq 0$:

$$E_n(0) = -\frac{1}{n+1}(2^{n+2} - 2) B_{n+1},$$

$$B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n}) B_n.$$

In addition, for even $n \geq 0$, the following can also be derived from (7)-(9):

$$B_n\left(\frac{1}{3}\right) = -\frac{1}{2}(1-3^{1-n})B_n,$$

$$B_n\left(\frac{1}{4}\right) = -2^{-n}(1-2^{1-n})B_n,$$

$$B_n\left(\frac{1}{6}\right) = \frac{1}{2}(1-2^{1-n})(1-3^{1-n})B_n,$$

$$E_n\left(\frac{1}{6}\right) = 2^{-n-1}(1+3^{-n})E_n.$$

Also, for odd $n \geq 1$, we have

$$E_n\left(\frac{1}{3}\right) = -\frac{1}{n+1}(2^{n+1}-1)(1-3^{-n})B_{n+1},$$

$$B_n\left(\frac{1}{4}\right) = -n4^{-n}E_{n-1}.$$

Each of these can be found in [11]. Similar expressions have been found for each of $B_n(1/3)$ and $B_n(1/6)$ when n is odd, but these are in terms of a sequence of rational values I_n, whose denominators consist of certain powers of 3 (see [5], [6]).

Bernoulli and Euler numbers and polynomials have numerous applications in mathematics. Because of this, they have been studied quite extensively. Besides the study of these polynomials at specific rational points, efforts have also been made to find congruence relations that describe specific Bernoulli polynomials at arbitrary rational points. A. Granville and Z.-W. Sun [7] have shown that if an integer $q \geq 3$ is odd and $1 \leq a \leq q$, with $(a, q) = 1$, then for p prime,

$$B_{p-1}\left(\frac{a}{q}\right) - B_{p-1} \equiv 2^{-1}p^{-1}q(U_p - 1) \pmod{p},$$

where U_p is a linear recurrence of order $[q/2]$ depending only on a, q, and the least positive residue of p modulo q. Their work extended a list of congruences given by E. Lehmer [9].

In this note we illustrate a means of finding congruence relations among Bernoulli and Euler polynomials evaluated at various rational numbers. We do this by considering the polynomials at values that have not been discussed previously. By applying (7)-(9), we build linear relationships among certain rational evaluations. Some recent results concerning the values of Bernoulli and Euler polynomials at rational points at rational points then enable us to obtain congruences based on the coefficients of these relations. Before proceeding with the derivation of the congruences, we shall present these results.

2. SOME RECENT RESULTS

The following result concerning Bernoulli polynomials was recently presented by G. Almkvist and A. Meurman in [1]. Other versions of the proof of this are given in [2], [3], and [13].

Theorem 2.1: Let $r, s \in \mathbb{Z}$, $s \neq 0$. Then $s^n(B_n(r/s) - B_n(0)) \in \mathbb{Z}$.

Since Euler polynomials satisfy many properties that are similar to those that Bernoulli polynomials satisfy, we would expect a result similar to Theorem 2.1 for Euler polynomials. In fact, we have such a result, presented in [4].

Theorem 2.2: Let $r, s \in \mathbb{Z}$, $s \neq 0$. Then $s^n(E_n(r/s) - (-1)^{rs}E_n(0)) \in \mathbb{Z}$.

Note that Theorems 2.1 and 2.2 will be the key components that enable us to derive the congruences that we intend to illustrate. They imply that, whenever k is a positive integer, for all $r, s \in \mathbb{Z}$, $s \neq 0$, the Bernoulli polynomials satisfy

$$ks^nB_n\left(\frac{r}{s}\right) \equiv ks^nB_n \pmod{k},$$

and for the Euler polynomials,

$$ks^nE_n\left(\frac{r}{s}\right) \equiv (-1)^{rs}ks^nE_n(0) \pmod{k}.$$

Note that this last congruence can be written in terms of B_n since we can also express $E_n(0)$ in such manner.

3. SOME EXAMPLES

The multiplication identities (7)-(9) provide a linear relationship among a set of values of particular Bernoulli and Euler polynomials at various rational numbers, these numbers also satisfying their own linear relationship. Varying the parameters t and q in (7)-(9) may provide several distinct linear relationships among these values. By a partial reduction of such a system, the coefficients of these values are modified so that, by applying Theorems 2.1 and 2.2, a congruence relationship can be obtained modulo one of these coefficients.

3.1 A Congruence Relating $B_n(2r/s)$ and $E_{n-1}(2r/s)$

This example gives a congruence relation, modulo a power of 2, between Bernoulli and Euler polynomials evaluated at the same rational number.

Theorem 3.1: Let $r, s \in \mathbb{Z}$ such that $(2r, s) = 1$. Then for positive integers n,

$$2B_n\left(\frac{2r}{s}\right) - nE_{n-1}\left(\frac{2r}{s}\right) \equiv 2^{n+1}B_n \pmod{2^{n+1}}.$$

Proof: Letting $q = 2$ and $t = r/s$ in (7) and (9) yields

$$B_n\left(\frac{2r}{s}\right) = 2^{n-1}B_n\left(\frac{r}{s}\right) + 2^{n-1}B_n\left(\frac{s+2r}{2s}\right),$$

$$-nE_{n-1}\left(\frac{2r}{s}\right) = 2^nB_n\left(\frac{r}{s}\right) - 2^nB_n\left(\frac{s+2r}{2s}\right).$$

Combining these two relations so as to eliminate $B_n((s+2r)/(2s))$, we obtain

$$2B_n\left(\frac{2r}{s}\right) - nE_{n-1}\left(\frac{2r}{s}\right) = 2^{n+1}B_n\left(\frac{r}{s}\right),$$

and thus, by Theorem 2.1,

$$2s^n B_n\left(\frac{2r}{s}\right) - ns^n E_{n-1}\left(\frac{2r}{s}\right) \equiv 2^{n+1} s^n B_n \pmod{2^{n+1}}.$$

This then yields the theorem, since $(2, s) = 1$. □

This result implies that, for odd $n \geq 3$, we have

$$2B_n\left(\frac{2r}{s}\right) \equiv nE_{n-1}\left(\frac{2r}{s}\right) \pmod{2^{n+1}},$$

since $B_n = 0$, and for all $n \geq 1$,

$$2B_n\left(\frac{2r}{s}\right) \equiv nE_{n-1}\left(\frac{2r}{s}\right) \pmod{2^n},$$

by the von Staudt-Clausen theorem.

3.2 Congruences for $B_n(t)$ and $E_{n-1}(t)$ at Multiples of $1/10$ for n Even

This additional example concerns the values of $B_n(t)$, at $1/10$ and $3/10$, and $E_{n-1}(t)$, at $1/5$ and $2/5$, for even $n \geq 2$.

Lemma 3.2: Let be an even positive integer. Then

$$10^n B_n\left(\frac{1}{10}\right) + 10^n B_n\left(\frac{3}{10}\right) = (2^{n-1} - 1)(5^n - 5)B_n, \tag{10}$$

$$n5^n E_{n-1}\left(\frac{1}{5}\right) - n5^n E_{n-1}\left(\frac{2}{5}\right) = -(2^n - 1)(5^n - 5)B_n, \tag{11}$$

$$(2^n + 1)10^n B_n\left(\frac{1}{10}\right) + (2^{n-1} + 1)n5^n E_{n-1}\left(\frac{1}{5}\right) = -2^{n-1}(5^n - 5)B_n, \tag{12}$$

$$2(2^n + 1)5^n B_n\left(\frac{1}{5}\right) + n5^n E_{n-1}\left(\frac{1}{5}\right) = -2^n(5^n - 5)B_n. \tag{13}$$

Proof: In view of (3), by letting $q = 2$ and $t = 1/5, 1/10$ in (7) and (9), we obtain

$$2^n \cdot 5^n B_n\left(\frac{1}{5}\right) - 2 \cdot 5^n B_n\left(\frac{2}{5}\right) + 10^n B_n\left(\frac{3}{10}\right) = 0, \tag{14}$$

$$2^n \cdot 5^n B_n\left(\frac{1}{5}\right) - 10^n B_n\left(\frac{3}{10}\right) + n5^n E_{n-1}\left(\frac{2}{5}\right) = 0, \tag{15}$$

$$-2 \cdot 5^n B_n\left(\frac{1}{5}\right) + 2^n \cdot 5^n B_n\left(\frac{2}{5}\right) + 10^n B_n\left(\frac{1}{10}\right) = 0, \tag{16}$$

$$-2^n \cdot 5^n B_n\left(\frac{2}{5}\right) + 10^n B_n\left(\frac{1}{10}\right) + n5^n E_{n-1}\left(\frac{1}{5}\right) = 0. \tag{17}$$

The case of $q = 5$ and $t = 0$ in (7), yields

$$5^n B_n\left(\frac{1}{5}\right) + 5^n B_n\left(\frac{2}{5}\right) = -\frac{1}{2}(5^n - 5)B_n. \tag{18}$$

Note that, by adding corresponding left-hand and right-hand sides of each equation, the combination $(14)+(16)-(2^n-2)(18)$ yields (10). Also, $(17)+2(2^n-1)(18)-(14)-(15)-(16)$ yields (11). From $2^{n-1}(16)+(2^{n-1}+1)(17)+2^n(18)$, we obtain (12), and $(17)+2^{n+1}(18)-(16)$ yields (13). □

Now, from (10)-(13), we can derive congruences related to each of the values $10^nB_n(1/10)$, $10^nB_n(3/10)$, $n5^nE_{n-1}(1/5)$, and $n5^nE_{n-1}(2/5)$. We shall first focus on those for the Euler polynomials.

Theorem 3.3: For n an even positive integer,

$$n5^nE_{n-1}\left(\frac{1}{5}\right)\equiv-(2^n(5^n-5)+5^n(2^{n+1}+2))B_n \pmod{2^{n+1}+2}, \tag{19}$$

$$n5^nE_{n-1}\left(\frac{2}{5}\right)\equiv-(5^n-5+5^n(2^{n+1}+2))B_n \pmod{2^{n+1}+2}. \tag{20}$$

Proof: If we use Theorem 2.1 to reduce (13) modulo $2^{n+1}+2$, we obtain (19). Now reduce (11) modulo $2^{n+1}+2$, utilizing (19) to represent $n5^nE_{n-1}(1/5)$, and we obtain (20). □

Corollary 3.4: Let n be an even positive integer, and let p be prime such that $p\,|\,(2^n+1)$. Then

$$n5^nE_{n-1}\left(\frac{1}{5}\right)\equiv-n5^nE_{n-1}\left(\frac{2}{5}\right)\equiv(5^n-5)B_n \pmod{p}.$$

Proof: If $p\,|\,(2^n+1)$, then $(p-1)\nmid n$ since, otherwise, $2^n+1\equiv 2 \pmod{p}$. Thus, by the von Staudt-Clausen theorem, p is not in the denominator of B_n, and so $5^n(2^{n+1}+2)B_n\equiv 0 \pmod{p}$. Therefore, (19) and (20) reduce to yield the result. □

Corollary 3.5: Let p be prime such that $p\equiv 5 \pmod 8$. Then

$$5^{(p-1)/2}E_{(p-3)/2}\left(\frac{1}{5}\right)\equiv-5^{(p-1)/2}E_{(p-3)/2}\left(\frac{2}{5}\right)\equiv-2(5^{(p-1)/2}-5)B_{(p-1)/2} \pmod{p}.$$

Proof: Note that $p\equiv 5 \pmod 8$ implies that $(p-1)/2$ is even and that $\left(\frac{2}{p}\right)=-1$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol corresponding to p. Euler's criterion states that $\left(\frac{2}{p}\right)=-1$ if and only if $2^{(p-1)/2}+1\equiv 0 \pmod{p}$. Therefore, by taking $n=(p-1)/2$, the result follows. □

Corollary 3.6: Let p be prime such that $p\equiv 13 \pmod{24}$. If there exist integers C and D for which $p=C^2+27D^2$, then

$$5^{(p-1)/6}E_{(p-7)/6}\left(\frac{1}{5}\right)\equiv-5^{(p-1)/6}E_{(p-7)/6}\left(\frac{2}{5}\right)\equiv-6(5^{(p-1)/6}-5)B_{(p-1)/6} \pmod{p}.$$

Proof: In [8], page 119, we see that there are integers C and D such that $p=C^2+27D^2$ if and only if 2 is a cubic residue modulo p. Now, 2 is a cubic residue modulo p if and only if $2^{(p-1)/3}\equiv 1 \pmod{p}$, and since $(p-1)/6$ must be an (even) integer, we can write

$$2^{(p-1)/3}-1=(2^{(p-1)/6}-1)(2^{(p-1)/6}+1),$$

where either $2^{(p-1)/6}\equiv-1 \pmod{p}$ or $2^{(p-1)/6}\equiv 1 \pmod{p}$.

If $2^{(p-1)/6}\equiv 1 \pmod{p}$, then $2^{(p-1)/2}\equiv 1 \pmod{p}$; thus, by Euler's criterion, $\left(\frac{2}{p}\right)=1$. However, $p\equiv 13 \pmod{24}$ implies that $\left(\frac{2}{p}\right)=-1$. Therefore, $2^{(p-1)/6}\equiv-1 \pmod{p}$, yielding the result. □

Now we consider congruences for $10^n B_n(1/10)$ and $10^n B_n(3/10)$.

Theorem 3.7: Let n be an even positive integer. Then

$$(2^n+1)10^n B_n\left(\frac{1}{10}\right) \equiv -(5^n(2^{2n}+2^n-2)+2^{n-1}(5^n-5))B_n \pmod{5n(2^{n-1}+1)}, \tag{21}$$

$$(2^n+1)10^n B_n\left(\frac{3}{10}\right) \equiv (5^n(2^{2n}+2^n-2)+(2^{2n-1}-1)(5^n-5))B_n \pmod{5n(2^{n-1}+1)}. \tag{22}$$

Proof: By Theorem 2.2, we can reduce (12) modulo $5n(2^{n-1}+1)$ to obtain

$$(2^n+1)10^n B_n\left(\frac{1}{10}\right) - (2^{n-1}+1)n5^n E_{n-1}(0) \equiv -2^{n-1}(5^n-5)B_n \pmod{5n(2^{n-1}+1)}.$$

Since $-nE_{n-1}(0) = (2^{n+1}-2)B_n$, this yields (21). Now multiply (10) through by 2^n+1 and reduce modulo $5n(2^{n-1}+1)$, utilizing (21) to represent $(2^n+1)10^n B_n(1/10)$, and we obtain (22). □

Corollary 3.8: Let p be prime, $p>3$, and let n be an even positive integer such that $p\,|\,(2^{n-1}+1)$. Then

$$10^n B_n\left(\frac{1}{10}\right) \equiv 10^n B_n\left(\frac{3}{10}\right) \equiv -(5^n-5)B_n \pmod{p}.$$

Proof: If $p\,|\,(2^n+2)$, then $(p-1)\nmid n$ since, otherwise, $2^n+2\equiv 3 \pmod{p}$. Thus, p is not in the denominator of B_n. This implies that we can reduce (21) to the form

$$(2^n+1)10^n B_n\left(\frac{1}{10}\right) \equiv (5^n-5)B_n \pmod{p}.$$

Also, from Theorem 2.1,

$$\begin{aligned}(2^n+1)10^n B_n\left(\frac{1}{10}\right) &\equiv (2^n+2)10^n B_n - 10^n B_n\left(\frac{1}{10}\right) \pmod{2^n+2}\\ &\equiv -10^n B_n\left(\frac{1}{10}\right) \pmod{p}.\end{aligned}$$

Thus, we have the congruence for $10^n B_n(1/10)$. By incorporating this into the reduction of (10) modulo p, we can obtain the congruence for $10^n B_n(3/10)$. □

Corollary 3.9: Let p be prime, $p>3$, such that $p\equiv 3 \pmod{8}$. Then

$$10^{(p+1)/2}B_{(p+1)/2}\left(\frac{1}{10}\right) \equiv 10^{(p+1)/2}B_{(p+1)/2}\left(\frac{3}{10}\right) \equiv -(5^{(p+1)/2}-5)B_{(p+1)/2} \pmod{p}.$$

Proof: If $p\equiv 3 \pmod{8}$, then $(p+1)/2$ is even and $\left(\frac{2}{p}\right)=-1$. By Euler's criterion, we then have $2^{(p-1)/2}+1\equiv 0 \pmod{p}$. The result follows by taking $n=(p+1)/2$. □

Corollary 3.10: Let p be prime such that $p\equiv 11$ or $19 \pmod{40}$. Then p divides the numerators of $B_{(p+1)/2}(1/10)$ and $B_{(p+1)/2}(3/10)$.

Proof: If $p\equiv 11$ or $19 \pmod{40}$, then $p\equiv 3 \pmod{8}$ and $\left(\frac{5}{p}\right)=1$. By Euler's criterion, $\left(\frac{5}{p}\right)=1$ implies that $5^{(p+1)/2}-5\equiv 0 \pmod{p}$. Since these conditions also imply that $p\nmid 10$, the result follows. □

Corollary 3.11: Let p be prime such that $p \equiv 19 \pmod{24}$. If there exist integers C and D for which $p = C^2 + 27D^2$, then

$$10^{(p+5)/6} B_{(p+5)/6}\left(\frac{1}{10}\right) \equiv 10^{(p+5)/6} B_{(p+5)/6}\left(\frac{3}{10}\right) \equiv -(5^{(p+5)/6} - 5) B_{(p+5)/6} \pmod{p}.$$

Proof: Recall that there are integers C and D such that $p = C^2 + 27D^2$ if and only if $2^{(p-1)/3} \equiv 1 \pmod{p}$. Since $(p-1)/6$ is an integer, this implies that either $2^{(p-1)/6} \equiv -1 \pmod{p}$ or $2^{(p-1)/6} \equiv 1 \pmod{p}$.

If $2^{(p-1)/6} \equiv 1 \pmod{p}$, then $2^{(p-1)/2} \equiv 1 \pmod{p}$; thus, $\left(\frac{2}{p}\right) = 1$. However, $p \equiv 19 \pmod{24}$ implies that $\left(\frac{2}{p}\right) = -1$. Therefore, $2^{(p-1)/6} \equiv -1 \pmod{p}$, and so $p \mid (2^{(p+5)/6} + 2)$. □

4. CONCLUSION

We have illustrated how some simple properties of Bernoulli and Euler polynomials can be utilized to construct congruences for certain rational evaluations of these polynomials. Congruences involving more terms can be easily obtained, but the difficulty to interpret their meaning increases with the number of terms involved. The examples given here are simple, but they are quite effective at illustrating how this method provides an opportunity to obtain previously unknown divisibility properties of rational values of Bernoulli and Euler polynomials.

REFERENCES

1. G. Almkvist & A. Meurman. "Values of Bernoulli Polynomials and Hurwitz's Zeta Function at Rational Points." *C. R. Math. Rep. Acad. Sci. Canada* **13** (1991):104-08.
2. K. Bartz & J. Rutkowski. "On the von Staudt-Clausen Theorem." *C. R. Math. Rep. Acad. Sci. Canada* **15** (1993):46-48.
3. F. Clarke & I. Sh. Slavutskii. "The Integrality of the Values of Bernoulli Polynomials and of Generalised Bernoulli Numbers." *Bull. London Math. Soc.* **29** (1997):22-24.
4. G. Fox. "Euler Polynomials at Rational Numbers." *C. R. Math. Rep. Acad. Sci. Canada* **21** (1999):87-90.
5. J. Glaisher. "The Bernoullian Function." *Quart. J. Pure Appl. Math.* **29** (1898):1-168.
6. J. Glaisher. "On a Set of Coefficients Analogous to the Eulerian Numbers." *Proc. Lond. Math. Soc.* **31** (1899):216-35.
7. A. Granville & Z.-W. Sun. "Values of Bernoulli Polynomials." *Pacific J. Math.* **172** (1996): 117-37.
8. K. Ireland & M. Rosen. *A Classical Introduction to Modern Number Theory*. New York: Springer-Verlag, 1982.
9. E. Lehmer. "On Congruences Involving Bernoulli Numbers and the Quotients of Fermat and Wilson." *Ann. of Math.* **39** (1938):350-60.
10. N. Nielsen. *Traité élémentaire des nombres de Bernoulli*. Paris, 1923.
11. N. E. Nörlund. *Vorlesungen über Differenzenrechnung*. Berlin: Springer-Verlag, 1924.
12. J. Raabe. "Zurückführung einiger Summen und bestimmten Integrale auf die Jacob-Bernoullische Function." *J. Reine Angew. Math.* **42** (1851):348-67.
13. B. Sury. "The Value of Bernoulli Polynomials at Rational Numbers." *Bull. London Math. Soc.* **25** (1993):327-29.

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