SETS IN WHICH THE PRODUCT OF ANY K ELEMENTS INCREASED BY t IS A k^{th}-POWER

Abdelkrim Kihel
Département de Mathématiques et de Statistique,
Université de Montréal, Montréal, Québec, Canada

Omar Kihel
Département de Mathématiques et de Statistique,
Pav. Vachon, Université Laval, G1K 7P4, Québec, Canada
(*Submitted January 1999-Final Revision October 1999*)

Let t be an integer. A P_t-set of size n is a set $A = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers such that $x_i x_j + t$ is a square of an integer whenever $i \neq j$. These P_t-sets are said to verify Diophantus' property. In fact, Diophantus was the first to note that the product of any two elements of the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ increased by 1 is a square of a rational number. We now introduce a more general definition.

Definition 1: Let $k > 1$ be a positive integer, and let t be an integer. A $P_t^{(k)}$-set of size n is a set $A = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers such that $\prod_{i \in I} x_i + t$ is a k^{th}-power of an integer for each $I \subset \{1, 2, \ldots, n\}$ where $\operatorname{card}(I) = k$.

A $P_t^{(k)}$-set A is said to be extendible if there exists an integer $a \notin A$ such that $A \cup \{a\}$ is a $P_t^{(k)}$-set. When $k = 2$, these sets are exactly the P_t-sets. The problem of extending P_t-sets is very old and dates back to the time of Diophantus (see Dickson [5], vol. II). The first famous result in this area is due to Baker and Davenport [3], who showed that the P_1-set $\{1, 3, 8, 120\}$ is nonextendible by using Diophantine approximation. Several others have recently made efforts to characterize the P_t-sets (see references). However, nothing is known about the $P_t^{(k)}$-sets when $k \geq 3$.

The purpose of this paper is to exhibit a $P_t^{(3)}$-set of size 4, and to show (Theorem 1) that this set is nonextendible. We also prove (Theorem 2) that the $P_{-8}^{(4)}$-set $\{1, 2, 3, 4\}$ and the $P_1^{(4)}$-set $\{1, 2, 5, 8\}$ are nonextendible. In Theorem 3 we show that any $P_t^{(k)}$-set is finite.

Example of a $P_t^{(3)}$-set: The set $\{1, 3, 4, 7\}$ is a $P_{-20}^{(3)}$-set of size 4.

Theorem 1: The $P_{-20}^{(3)}$-set $\{1, 3, 4, 7\}$ is nonextendible.

Proof: Suppose there exists an integer a such that $\{1, 3, 4, 7, a\}$ is a $P_{-20}^{(3)}$-set. Then the following system of equations

$$\begin{cases} 3a - 20 = u^3, \\ 21a - 20 = v^3, \\ 12a - 20 = w^3, \end{cases} \tag{1}$$

has an integral solution $(u, v, w) \in \mathbb{N}^3$. One can derive more equations in the system (1) but this is not necessary for our proof. The system (1) yields

$$u^3 + v^3 = 2w^3 \text{ with } (u, v, w) \in \mathbb{N}^3. \tag{2}$$

However, it is well known from the work of Euler and Lagrange (see Dickson [5], vol. II, pp. 572-73) that all solutions of equation (2) in positive integers are given by $u=v=w$, which is impossible in the system (1). □

It would be interesting to know if there exists any $P_t^{(k)}$-set of size $n>k\geq 4$. For $n=k$, the problem is easy. In fact, there are two strategies for finding a $P_t^{(k)}$-set of size k.

(1) Fix any k positive integers $a_1, a_2, \ldots, a_k$. Let A be an integer and $t = A^k - \prod_{i=1}^k a_i$. Then the set $\{a_1, a_2, \ldots, a_k\}$ is a $P_t^{(k)}$-set of size k. For example, let $k=4$, $a_1=1$, $a_2=2$, $a_3=3$, $a_4=4$, and $A=2$. Then $t=-8$ and $\{1, 2, 3, 4\}$ is a $P_{-8}^{(4)}$-set of size 4.

(2) Fix any t, and choose any integer A such that there exist k different factors $a_1, a_2, \ldots, a_k$ nonnecessary primes and $A^k - t = \prod_{i=1}^k a_i$. Then the set $\{a_1, a_2, \ldots, a_k\}$ is a $P_t^{(k)}$-set of size k. For example, let $k=4$, $t=1$, and $A=2$. Then $A^4 - t = 80 = 1\cdot 2\cdot 5\cdot 8$ and $\{1, 2, 5, 8\}$ is a $P_1^{(4)}$-set of size 4.

Theorem 2:

(a) The $P_{-8}^{(4)}$-set $\{1, 2, 3, 4\}$ is nonextendible.

(b) The $P_1^{(4)}$-set $\{1, 2, 5, 8\}$ is nonextendible.

Proof:

(a) Suppose there exists an integer a such that $\{1, 2, 3, 4, a\}$ is a $P_{-8}^{(4)}$-set. Then the following system of equations

$$\begin{cases} 6a-8=x^4, \\ 8a-8=y^4, \\ 12a-8=z^4, \\ 24a-8=w^4, \end{cases} \tag{3}$$

has an integral solution $(x, y, z, w) \in \mathbb{N}^4$. A congruence mod 16 shows that this is impossible.

(b) Suppose there exists an integer a such that $\{1, 2, 5, 8, a\}$ is a $P_1^{(4)}$-set. Then the following system of equations

$$\begin{cases} 10a+1=x^4, \\ 16a+1=y^4, \\ 40a+1=z^4, \\ 80a+1=w^4, \end{cases} \tag{4}$$

has an integral solution $(x, y, z, w) \in (\mathbb{N}^*)^4$. The system (4) yields

$$w^4+1=2z^4 \text{ with } (z, w) \in (\mathbb{N}^*)^2. \tag{5}$$

But it is well known (see [13], pp. 17-18) that all solutions of (5) are given by $w=z=1$, and this gives $a=0$. □

Theorem 3: Any $P_t^{(k)}$-set is finite.

Proof: Let $\{a_1, \ldots, a_k, a_{k+1}, N\}$ be a $P_t^{(k)}$-set. Let $a = a_1a_2\ldots a_ka_{k+1}$,

$$\alpha = \frac{a}{a_1a_2}, \quad \beta = \frac{a}{a_1a_3}, \quad \text{and} \quad \gamma = \frac{a}{a_2a_3}.$$

Then there exist integers x, y, and z such that

$$\alpha N+t=x^k,\quad \beta N+t=y^k,\quad \text{and}\quad \gamma N+t=z^k.$$

Hence, we obtain a superelliptic curve

$$(\alpha N+t)(\beta N+t)(\gamma N+t)=w^k$$

(for $k=2,3$, this is an elliptic curve), and from Theorems 6.1 and 6.2 in [15] it follows that $N\le C$ for some computable number C depending only on k, α, β, γ, and t. □

ACKNOWLEDGMENT

The authors are extremely grateful to the anonymous referee for a very careful reading of the manuscript and for many suggestions that led to an improved presentation of this paper.

REFERENCES

1. J. Arkin, V. E. Hoggatt, Jr., & E. G. Strauss. "On Euler's Solution of a Problem of Diophantus." *The Fibonacci Quarterly* **17.4** (1979):333-39.
2. J. Arkin, V. E. Hoggatt, Jr., & E. G. Strauss. "On Euler's Solution of a Problem of Diophantus II." *The Fibonacci Quarterly* **18.2** (1980):170-76.
3. A. Baker & H. Davenport. "The Equations $3x^2-2=y^2$ and $8x^2-7=z^2$." *Quart. J. Math. Oxford*, Ser. (2), **20** (1969):129-37.
4. E. Brown. "Sets in Which $xy+k$ Is Always a Square." *Math. Comp.* **45** (1985):613-20.
5. L. E. Dickson. *History of the Theory of Numbers*. Vol. 2. New York: Chelsea, 1966.
6. A. Dujella. "Generalization of a Problem of Diophantus." *Acta Arith.* **65** (1993):15-27.
7. A. Dujella. "On Diophantine Quintuples." *Acta Arith.* **81** (1997):69-79.
8. A. Dujella. "An Extension of an Old Problem of Diophantus and Euler." *The Fibonacci Quarterly* **37.4** (1999):312-14.
9. A. Dujella & A. Pethoe. "Generalization of a Theorem of Baker and Davenport." *Quart. J. Math. Oxford*, Ser. (2), **49** (1998):291-306.
10. P. Heichelheim. "The Study of Positive Integers (a,b) Such That $ab+1$ Is a Square." *The Fibonacci Quarterly* **17.3** (1979):269-74.
11. B. W. Jones. "A Variation on a Problem of Davenport and Diophantus." *Quart. J. Math. Oxford*, Ser (2), **27** (1976):349-53.
12. B. W. Jones. "A Second Variation on a Problem of Diophantus and Davenport." *The Fibonacci Quarterly* **16.2** (1978):155-65.
13. L. J. Mordell. *Diophantine Equations*. London: Academic Press, 1969.
14. V. K. Mootha & G. Berzsenyi. "Characterization and Extendibility of P_t-Sets." *The Fibonacci Quarterly* **27.3** (1989):287-88.
15. T. N. Shorey & R. Tijdeman. *Exponential Diophantine Equations*. Cambridge: Cambridge University Press, 1986.

AMS Classification Number: 11D25

❖❖❖