REMARKS ON THE "GREEDY ODD" EGYPTIAN FRACTION ALGORITHM

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1. INTRODUCTION

We denote the set of positive integers by $\mathbf{N}$. Consider $a, b \in \mathbf{N}$ with

$$a < b, \quad (a, b) = 1. \tag{1.1}$$

Fibonacci, in 1202 ([8], see also [1], [7]) introduced the *greedy algorithm*: we take the greatest Egyptian fraction $1/x_1$ with $1/x_1 \le a/b$, form the difference $a/b - 1/x_1 =: a_1/b_1$ [where $(a_1, b_1) = 1$] and, if a_1/b_1 is not zero, continue similarly. It is easily seen that the sequence of numerators $a_0 := a, a_1, a_2, \ldots$ is strictly decreasing, from which it follows that after finitely many, say s, steps $(s \le a)$, the process will stop. This gives us a representation

$$\frac{a}{b} = \frac{1}{x_1} + \cdots + \frac{1}{x_s}, \quad 1 < x_1 < \cdots < x_s. \tag{1.2}$$

If b is *odd*, the *greedy odd algorithm* is defined as follows: we take the greatest Egyptian fraction $1/x_1$ with x_1 odd, $1/x_1 \le a/b$, and continue similarly. We have (see [4], [3], [5]) the interesting

Open Problem 1.1: Does the greedy odd algorithm (for b odd) always stop after finitely many steps?

In this paper, using elementary methods, we study some properties of the greedy odd algorithm. In Section 2 we fix the notation and record some obvious facts. In Section 3, the main part of this paper, we prove some results on the possibility of occurrence of certain initial sequences of the sequence $a_0 := a, a_1, a_2, \ldots$ of numerators connected with the greedy odd algorithm. We hope that at least some of our results are new.

2. THE GREEDY ODD ALGORITHM

We suppose that in (1.1) b is *odd* and sometimes we write $b = 2k+1$, where $k \in \mathbf{N}$. Now, since only odd denominators are used in the Egyptian fractions, we agree to write $x = 2n+1$, where $n \in \mathbf{N}$. To start the greedy odd algorithm, we take the unique $n_1 \in \mathbf{N}$ satisfying the condition

$$\frac{1}{2n_1+1} \le \frac{a}{2k+1} < \frac{1}{2n_1-1}, \tag{2.1}$$

and then we write

$$\frac{a}{2k+1} - \frac{1}{2n_1+1} =: \frac{a_1'}{(2k+1)(2n_1+1)} =: \frac{a_1}{2k_1+1} \text{ with } (a_1, 2k_1+1) = 1. \tag{2.2}$$

Case A) If

$$\frac{a}{2k+1} \in \left[\frac{1}{2n_1+1}, \frac{1}{2n_1}\right),$$

then $0 \le a_1' < a$ and so $0 \le a_1 < a$ (this case corresponds to the normal greedy algorithm).

Case B) If

$$\frac{a}{2k+1} \in \left[\frac{1}{2n_1}, \frac{1}{2n_1-1}\right),$$

then it is easily seen that $a < a_1' < 2a$.

Case B1) If $d := (a_1', (2k+1)(2n_1+1)) > 1$, we cancel and find that $0 < a_1 < a$. (In fact, as d is odd, $d \ge 3$, and therefore $a_1' = da_1 < 2a$ implies that $a_1 < 2a/d \le 2a/3 < a$.)

Case B2) If $d = 1$, then $a_1 = a_1'$ and so $a < a_1 < 2a$.

We find that A) and B1) are "good" cases (numerator decreases), while B2) is a "bad" case (numerator increases).

We form the sequence of numerators $a_0 := a, a_1, a_2, \ldots$. If $a_s = 0$ for some $s \in \mathbf{N}$, then the greedy odd algorithm stops and we get

$$\frac{a}{b} = \frac{a}{2k+1} = \frac{1}{x_1} + \cdots + \frac{1}{x_s} \tag{2.3}$$

with $x_1 = 2n_1+1, \ldots, x_s = 2n_s+1$. It is clear that $a_s = 0$ if and only if $a_{s-1} = 1$.

From (2.2), it follows immediately that

$$a_i \not\equiv a_{i+1} \pmod 2 \text{ for } i = 0, 1, 2, \ldots. \tag{2.4}$$

Example 2.1: The sequence of numerators $a_0, \ldots, a_{s-1}$ with $s = 19$, corresponding to the greedy odd algorithm for the fraction 5/139, is

$$5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26, 51, 2, 3, 4, 1.$$

Here, all cases are either B1) (occurs two times) or B2). The reader can find more examples in [4] (see also Examples 3.9 below).

Remark 2.2: Take any $a \in \mathbf{N}$, $a > 1$. Then take any $b \in \mathbf{N}$, b odd, such that (1.1) holds and form the sequence of numerators $a_0 := a, a_1, a_2, \ldots$ connected with the greedy odd algorithm for the fraction a/b. The question "Does 1 occur in this sequence?" is equivalent to Open Problem 1.1 and shows some similarity to the well-known (or "infamous" [4]) "$3x+1$"-problem (see, e.g., [6]).

If the greedy odd algorithm for the fraction a/b stops with $a_s = 0$, we write

$$h\left(\frac{a}{b}\right) = h\left(\frac{a}{2k+1}\right) := s$$

for the *number of steps*. Otherwise, we write

$$h\left(\frac{a}{2k+1}\right) := \infty.$$

If $h(a/b) < \infty$, then a trivial consequence of (2.3) is that

$$h(a/b) \equiv a \pmod 2. \tag{2.5}$$

Theorem 2.3: Suppose that $s \in \mathbb{N}$ is given. There are infinitely many fractions a/b, b odd, satisfying (1.1) such that

$$h(a/b) = s. \tag{2.6}$$

Proof: Take any sequence $x_1, \ldots, x_s$ satisfying

$$x_1 > 1,\ x_{i+1} \geq x_i^2 - x_i + 1,\ i = 1, \ldots, s-1, \tag{2.7}$$

such that x_i is odd for $i = 1, \ldots, s$. (For example, we can take $x_1 := 2n+1$ for $n \in \mathbb{N}$ and then define $x_{i+1} := x_i^2 - x_i + 1$ for $i = 1, \ldots, s-1$.) According to a result of J. J. Sylvester [9], the right-hand side of the definition $a/b := 1/x_1 + \cdots + 1/x_s$ is the result of applying the normal greedy algorithm to the fraction a/b. Note that b is odd since $x_1, \ldots, x_s$ are all odd. (We take, of course, $(a, b) = 1$.) Moreover, $a < b$. (We have, in fact, $a/b < 1/2$, see (2.9) below.) But it is obvious that, if the normal greedy algorithm produces only odd denominators $x_1, \ldots, x_s$, then the greedy odd algorithm, applied to a/b, is identical to the normal greedy algorithm (all cases are A)), and so $h(a/b) = s$. Since different sequences $x_1, \ldots, x_s$ satisfying (2.7) produce different fractions a/b, and since we have indicated how to choose infinitely many such sequences (with all x_i odd), the theorem follows. □

We close this section with two remarks.

Remark 2.4: We saw in (1.2) that $x_2 > x_1$ for the normal greedy algorithm (supposing, of course, that $a > 1$). It is easily seen that, in the case of the greedy odd algorithm for the fraction a/b, b odd, satisfying (1.1), the only possibility for $x_2 = x_1$ is $x_1 = x_2 = 3$, and it occurs if and only if

$$\frac{2}{3} \leq \frac{a}{b}. \tag{2.8}$$

For example, the greedy odd algorithm gives

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3},\ \frac{4}{5} = \frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{45},\ \frac{5}{7} = \frac{1}{3} + \frac{1}{3} + \frac{1}{21},\ \text{and}\ \frac{6}{7} = \frac{1}{3} + \frac{1}{3} + \frac{1}{7} + \frac{1}{21}.$$

Remark 2.5: If (1.1) holds and b is *even*, then it is clear that the greedy odd algorithm never stops. It is easily seen, for example, that

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \cdots = \sum_{i=1}^{\infty} \frac{1}{x_i}, \tag{2.9}$$

where $x_1 := 3$, $x_{i+1} := x_i^2 - x_i + 1$ for $i = 1, 2, \ldots$, is the result of applying the greedy odd algorithm to the fraction $1/2$. The equation (2.9) is, of course, well known (indeed, "famous" [2]).

3. ON SOME INITIAL SEQUENCES OF NUMERATORS

We are interested in the case B) of the greedy odd algorithm (see the beginning of Section 2). We suppose that $a > 1$ and $a < c < 2a$ with $c \in \mathbb{N}$. We search for odd $b = 2k + 1$, such that in the first step of the greedy odd algorithm for the fraction a/b, we have $c = a_1'$ (see (2.2)). Here we must suppose that $a \not\equiv c \pmod 2$ (see (2.4)).

Theorem 3.1: Let $a > 1$ and $a < c < 2a$, where $1 + c - a := 2t$ ($t \in \mathbb{N}$). Let $k \in \mathbb{N}$ satisfy $k + t \equiv 0 \pmod a$ and $(a, 2k+1) = 1$. Take $n_1 := (k+t)/a \in \mathbb{N}$. Then $a < 2k+1$, and in the first step of the greedy odd algorithm for the fraction $a/(2k+1)$, we have the case B). Moreover, $x_1 = 2n_1+1$, and

$$\frac{a}{2k+1} - \frac{1}{2n_1+1} = \frac{c}{(2k+1)(2n_1+1)}. \tag{3.1}$$

Proof: We have, by assumption,

$$2t = 1+c-a < 1+2a-a = 1+a. \tag{3.2}$$

Since $k+t \equiv 0 \pmod{a}$, we have $k+t \geq a$, implying by (3.2) that $k \geq a-t > a-(a+1)/2 = (a-1)/2$. Therefore, $a < 2k+1$.

We prove next that

$$\left\lceil \frac{2k+1}{a} \right\rceil = 2n_1, \tag{3.3}$$

from which it immediately follows that $x_1 = 2n_1+1$.

We have $2n_1 = (2k+2t)/a = (2k+1)/a + (2t-1)/a$, where, by (3.2), $0 < (2t-1)/a < 1$. This proves (3.3).

A simple calculation shows that (3.1) holds. Since $a_1' = c > a$, we must have the case B). □

Remark 3.2: In some cases, it is impossible to satisfy the condition $(a, 2k+1) = 1$ in Theorem 3.1. Take, for example, $a := 6$ and $c := 9$. Now $1+c-a = 1+9-6 = 4 =: 2t$, so that $t = 2$. If $k+2 \equiv 0 \pmod{6}$, then $2k+1 \equiv 3 \pmod{6}$, and so $(a, 2k+1) = 3$.

Corollary 3.3: Let $a > 1$ and $c := a+1$. If $k+1 \equiv 0 \pmod{a}$, then $(a, 2k+1) = 1$ and, for $n_1 := (k+1)/a$, the conclusions of Theorem 3.1 hold.

Proof: In this case, $1+c-a = 2 =: 2t$, so that $t = 1$. We need only prove that $(a, 2k+1) = 1$. This follows immediately from $2k+1 = 2(n_1 a - 1)+1 = 2n_1 a - 1$. □

For the rest of this paper, we consider the following problem. Let $a > 1$ be given. We search for such numbers $k \in \mathbf{N}$ that the greedy odd algorithm, applied to the fraction $a/(2k+1)$, starts with some cases B2) in such a way that the sequence of numerators $a_0 := a, a_1, a_2, \ldots$ starts with $a, a+1, a+2, \ldots$. Our main tool is Corollary 3.3 and our main achievement (see Theorem 3.7) is the following. For any $a > 1$, we give explicitly infinitely many numbers k such that the greedy odd algorithm for the fraction $a/(2k+1)$ starts with two cases B2), the numerator increasing by one in both steps.

Suppose now that we have used Corollary 3.3 once and that the first step corresponded to the case B2). We consider the fraction

$$\frac{a_1}{(2k+1)(2n_1+1)} =: \frac{a_1}{2k_1+1}, \text{ where } a_1 := a+1,$$

and use Corollary 3.3 again. Now,

$$k_1 = \frac{(2k+1)(2n_1+1)-1}{2} = \frac{4kn_1+2k+2n_1}{2} = 2kn_1+k+n_1,$$

so that we should have

$$2kn_1+k+n_1+1 \equiv 0 \pmod{a+1}. \tag{3.4}$$

But $2kn_1+k+n_1+1=2(n_1a-1)n_1+(n_1a-1)+n_1+1=2n_1^2a+n_1a-n_1=n_1(2n_1a+a-1)$ and $2n_1a+a-1=(a+1)+2k$, so that (3.4) will be satisfied if $k\equiv 0$ (mod $a+1$). We use the Chinese Remainder Theorem to solve the pair of simultaneous congruences

$$\begin{cases} k\equiv -1 \pmod{a} \\ k\equiv 0 \pmod{a+1} \end{cases}$$

and get the (unique) solution $k\equiv -(a+1)$ (mod $a(a+1)$). Now let $n_2:=(k_1+1)/(a+1)$. Then both steps correspond to the case B2) if and only if the conditions

(I) $$(a+1,(2k+1)(2n_1+1))=1 \text{ and}$$

(II) $$(a+2,(2k_1+1)(2n_2+1))=1$$

hold.

Lemma 3.4: Let $k:=-(a+1)+ja(a+1)$ with $j\in\mathbf{N}$. Condition (I) holds for every $j\in\mathbf{N}$.

Proof: We have $2k+1=(a+1)(2ja-2)+1$ (of course, $2k+1\equiv 1$ (mod $a+1$) since $k\equiv 0$ (mod $a+1$)) and therefore $(a+1,2k+1)=1$. We have $n_1=(k+1)/a=j(a+1)-1$, so $2n_1+1=2j(a+1)-1$ and therefore $(a+1,2n_1+1)=1$. It follows that (I) holds for every $j\in\mathbf{N}$. □

We have $2k+1=(a+2)(2ja-2-2j)+4j+3$, $2n_1+1=(a+2)2j-(2j+1)$, and

$$2n_2+1=(a+2)(4j^2a-4j^2-6j)+(2j+1)(4j+3), \tag{3.5}$$

from which it follows, since $2k_1+1=(2k+1)(2n_1+1)$, that

$$\text{(II) holds if and only if } (a+2,(2j+1)(4j+3))=1. \tag{3.6}$$

Theorem 3.5: Let $a>1$ and define k by $k:=-(a+1)+ja(a+1)$, $j\in\mathbf{N}$. A necessary and sufficient condition for the greedy odd algorithm for all the fractions $a/(2k+1)$, $j=1,2,\ldots$, to start with two cases B2), the numerators increasing by one, is that $a=2^r-2$, $r\geq 2$.

Proof: 1° Suppose that $a=2^r-2$ for some $r\in\mathbf{N}$, $r\geq 2$. By Lemma 3.4, (I) holds for every $j\in\mathbf{N}$. Condition (II) is now trivially satisfied, since $a+2=2^r$ and $(2k_1+1)(2n_2+1)$ is odd.

2° Suppose that $a\notin\{2^r-2: r\in\mathbf{N}, r\geq 2\}$. Then there exists an odd prime p such that $p\,|\,(a+2)$. Let j be such that $p=2j+1$. Then $p\,|\,(a+2,(2j+1)(4j+3))$, so that, by (3.6), (II) is not valid. □

By a similar argument, we can show the existence of certain short sequences of numerators of the form $a, a+1, 1$, where one case B2) and one case B1) are involved. More precisely, we have

Theorem 3.6: Let $a>1$ be odd. Let

$$k:=-\left(\frac{a^3+4a^2+5a+2}{2}\right)+ha(a+1)(a+2), \text{ where } h=1,2,\ldots. \tag{3.7}$$

Then the sequence of numerators, corresponding to the greedy odd algorithm for all the fractions $a/(2k+1)$, is $a, a+1, 1$.

Proof: We write $k:=-(a+1)+ja(a+1)$ with $j\in\mathbf{N}$ as before. By Lemma 3.4, (I) holds for every $j\in\mathbf{N}$. By assumption, $a+2$ is odd, so we can find $j_0\in\mathbf{N}$ such that $2j_0+1:=a+2$. If $j\equiv j_0$ (mod $a+2$), then $2j+1\equiv 0$ (mod $a+2$), and so, by (3.5), $2n_2+1\equiv 0$ (mod $a+2$). It follows that $(a+2, (2k_1+1)(2n_2+1)) = a+2$, implying $a_2 = 1$, for all such j. Writing $j := j_0 + (h-1)(a+2)$, we get (3.7). □

Theorem 3.5 gives, for some special numbers a, infinitely many numbers k such that the greedy odd algorithm, applied to the fraction $a/(2k+1)$, behaves in a certain manner. The first part of the next theorem is completely general, but the form of the numbers k is slightly more complicated.

Theorem 3.7: Let $a > 1$ and define k by $k := -(a+1)^2 + ha(a+1)(a+2)$, $h = 1, 2, \ldots$. Then the greedy odd algorithm for all the fractions $a/(2k+1)$ starts with two cases B2), the numerators increasing by one. Moreover, if $a = 2^r - 3$ $(r \geq 3)$, then the same holds for the third step.

Proof: We consider $k := -(a+1) + ja(a+1)$, $j \in \mathbf{N}$. By Lemma 3.4, (I) holds for every $j \in \mathbf{N}$. If $j \equiv -1 \pmod{a+2}$, then $2j+1 \equiv 4j+3 \equiv -1 \pmod{a+2}$, and therefore, by (3.6), condition (II) holds. Writing $j := -1 + h(a+2)$ with $h \in \mathbf{N}$, we get the first part of the theorem.

Consider now the third step. Defining $(2k_1+1)(2n_2+1) =: 2k_2+1$, we should have

$$k_2 + 1 \equiv 0 \pmod{a+2} \tag{3.8}$$

and then we will take $n_3 := (k_2+1)/(a+2)$. By a straightforward calculation,

$$\begin{aligned}\frac{k_2+1}{a+2} &= (-1+h+ah)\cdot(-1-2a+4ah+2a^2h)\cdot(4+15a+14a^2+4a^3-4h-34ah \\ &\quad -60a^2h-38a^3h-8a^4h+16ah^2+48a^2h^2+52a^3h^2+24a^4h^2+4a^5h^2),\end{aligned}$$

proving (3.8). The last part of the theorem follows now exactly as in the proof of Theorem 3.5. □

Taking $a := 2$, starting from Theorem 3.5, and using Corollary 3.3 two times, we obtained the following result.

Theorem 3.8: Let $k := 180g - 51$, $g = 1, 2, \ldots$. The greedy odd algorithm for all the fractions $2/(2k+1)$ starts with four cases B2), the numerators increasing by one.

Since we have suppressed the "dirty" details, we would like to give some examples of Theorem 3.8.

Examples 3.9: Using ten smallest values of g in Theorem 3.8, we get the following fractions with corresponding sequences of numerators (which should all start with 2, 3, 4, 5, 6).

$\frac{2}{259}$; 2, 3, 4, 5, 6, 1. $\frac{2}{619}$; 2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.

$\frac{2}{979}$; 2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1. $\frac{2}{1339}$; 2, 3, 4, 5, 6, 7, 2, 1.

$\frac{2}{1699}$; 2, 3, 4, 5, 6, 7, 8, 9, 10, 1. $\frac{2}{2059}$; 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 2, 1.

$\frac{2}{2419}$; 2, 3, 4, 5, 6, 1. $\frac{2}{2779}$; 2, 3, 4, 5, 6, 1.

$\frac{2}{3139}$; 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1. $\frac{2}{3499}$; 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1.

Using $g := 19$, we get

$$\frac{2}{6739}; 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 1.$$

We notice here that different fractions may have the same sequence of numerators connected with the greedy odd algorithm. The corresponding representations (2.3) are, of course, all different. Here, in the case of Theorem 3.8, we may note that, according to Corollary 3.3, $x_1 = k+2$. The first fraction $2/259$, for example, has the representation (2.3) with $s = 6$, where

$$\begin{aligned}
x_1 &= 131,\\
x_2 &= 11311,\\
x_3 &= 95942731,\\
x_4 &= 7364006009447959,\\
x_5 &= 451904870893213706499705982 73443,\\
x_6 &= 17504401057458184168608539983764625446136867132785710573437 90199.
\end{aligned}$$

We remark, finally, that the sequence of numerators $2, 3, 4, 5, 6, 1$ occurs whenever $g \equiv 0, 1 \pmod 7$ in Theorem 3.8.

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