ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-921 ***Proposed by the editors***

Determine whether or not $F_{6n} - 1$ and $F_{6n-3} + 1$ are relatively prime for all $n \geq 1$.

B-922 ***Proposed by Irving Kaplansky, Math. Sciences Research Institute, Berkeley, CA***

Determine all primes p such that the Fibonacci numbers modulo p yield all residues.

B-923 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let α_l be the l^{th} convergent of the continued fractional expansion:

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Prove that

(a) $\displaystyle \frac{1}{n}\sum_{k=0}^{n-1} \alpha_{l+k} \geq [F_n \alpha_l + F_{n-1}]^{1/n}$,

(b) $\displaystyle \alpha_k^l = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{\alpha_{l-1}^j}$ for all $k \in \mathbb{N}$.

B-924 ***Proposed by N. Gauthier, Royal Military College of Canada***

For n an arbitrary integer, the following identity is easily established for Lucas numbers:

$$L_{2n+2} + L_{2n-2} = 3L_{2n}. \tag{1}$$

Consider the Fibonacci and Lucas polynomials, $\{F_n(u)\}_{n=0}^{\infty}$ and $\{L_n(u)\}_{n=0}^{\infty}$, defined by

$$F_0(u) = 0,\ F_1(u) = 1,\ F_{n+2}(u) = uF_{n+1}(u) + F_n(u),$$
$$L_0(u) = 2,\ L_1(u) = u,\ L_{n+2}(u) = uL_{n+1}(u) + L_n(u),$$

respectively. The corresponding generalization of (1) is

$$L_{2n+1}(u) + L_{2n-2}(u) = (u^2+2)L_{2n}(u). \tag{2}$$

For m a nonnegative integer, with the convention that a discrete sum with a negative upper limit is identically zero, prove the following generalization of (2):

$$\begin{aligned}&(n+1)^{2m}L_{2n+2}(u) + (n-1)^{2m}L_{2n-2}(u)\\ &= (u^2+2)\left[\sum_{l=0}^{m}\binom{2m}{2l}n^{2l}\right]L_{2n}(u) + u(u^2+4)\left[n\sum_{l=0}^{m-1}\binom{2m}{2l+1}n^{2l}\right]F_{2n}(u).\end{aligned} \tag{3}$$

Also prove the following companion identity:

$$\begin{aligned}&(n+1)^{2m+1}F_{2n+2}(u) + (n-1)^{2m+1}F_{2n-2}(u)\\ &= u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}n^{2l}\right]L_{2n}(u) + (u^2+2)\left[\sum_{l=0}^{m-1}\binom{2m+1}{2l+1}n^{2l+1}\right]F_{2n}(u).\end{aligned} \tag{4}$$

SOLUTIONS

Determine the Determinant

B-906 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 38, no. 5, November 2000)***

Consider the following $n \times n$ determinants,

$$\Delta_1(n) := \begin{vmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix};$$

$$\Delta_2(n) := \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix};$$

n is taken to be a positive integer and $\Delta_1(0)=1$, $\Delta_2(0)=0$, by definition. Prove the following:

a. $\Delta_1(n)=F_{2n+1}$;

b. $\Delta_2(n)=F_{2n}$.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

It is rather obvious that $\Delta_2(n)=\Delta_3(n-1)$, where

$$\Delta_3(n):=\begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & 3 & -1 & \cdots & 0 & 0 & 0\\ 0 & -1 & 3 & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & -1 & 3 & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}_{n\times n}.$$

Expansion along the first row gives $\Delta_3(n)=3\Delta_3(n-1)-\Delta_3(n-2)$, thus

$$\Delta_2(n)=3\Delta_2(n-1)-\Delta_2(n-2).$$

On the other hand, $F_{2n}-F_{2n-2}=F_{2n-1}=F_{2n-2}+F_{2n-3}=2F_{2n-2}-F_{2n-4}$ implies that

$$F_{2n}=3F_{2(n-1)}-F_{2(n-2)}.$$

Hence, F_{2n} and $\Delta_2(n)$ satisfy the same recurrence relation. Since they have the same initial conditions, we conclude that $\Delta_2(n)=F_{2n}$. Finally, expansion along the first row of $\Delta_1(n)$ yields

$$\Delta_1(n)=2\Delta_3(n-1)-\Delta_2(n-1)=2F_{2n}-F_{2n-2}=F_{2n}+F_{2n-1}=F_{2n+1}.$$

Also solved by Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, Steve Edwards, Pentti Haukkanen, Steve Hennagin, Walther Janous, Carl Libis, Reiner Martin, John F. Morrison, Jaroslav Seibert, H.-J. Seiffert, James A. Sellers, Indulis Strazdins, and the proposer.

Fibonacci Bases and Exponents

B-907 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia (Vol. 38, no. 5, November 2000)***

Prove that

$$F_1^{F_1}\cdot F_2^{F_2}\cdot F_3^{F_3}\cdot\ldots\cdot F_n^{F_n}\le e^{(F_n-1)(F_{n+1}-1)}.$$

Solution by Reiner Martin, New York, NY

Recall that $\sum_{i=1}^n F_i=F_{n+2}-1$ and $\sum_{i=1}^n F_i^2=F_nF_{n+1}$. By the concavity of the logarithm, we get

$$\sum_{i=1}^n \frac{F_i}{F_{n+2}-1}\log F_i\le\log\left(\sum_{i=1}^n\frac{F_i^2}{F_{n+2}-1}\right)=\log\left(\frac{F_nF_{n+1}}{F_{n+2}-1}\right)$$
$$\le\frac{F_nF_{n+1}}{F_{n+2}-1}-1=\frac{(F_n-1)(F_{n+1}-1)}{F_{n+2}-1}.$$

Thus,

$$\sum_{i=1}^n F_i\log F_i\le(F_n-1)(F_{n+1}-1).$$

Taking exponentials yields the result.

Also solved by Charles Ashbacher, Paul S. Bruckman, Charles K. Cook, L. A. G. Dresel, Pentti Haukkanen, Walther Janous, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

A Fibonacci Polynomial Identity

B-908 ***Proposed by Indulis Strazdins, Riga Technical University, Latvia (Vol. 38, no. 5, November 2000)***

The Fibonacci polynomials, $F_n(x)$, are defined by

$$F_0(x)=0,\ F_1(x)=1,\ \text{and}\ F_{n+2}(x)=xF_{n+1}(x)+F_n(x)\ \text{for}\ n\geq 0.$$

Prove the identity

$$F_{n+1}^2(x)-4xF_n(x)F_{n-1}(x)=x^2F_{n-2}^2(x)+(x^2-1)F_{n-1}(x)(xF_n(x)-F_{n-3}(x)).$$

Solution by Maitland A. Rose, University of South Carolina, Sumter, SC

$$\begin{aligned}
&F_{n+1}^2(x)-4xF_n(x)F_{n-1}(x)=(xF_n(x)+F_{n-1}(x))^2-4xF_n(x)F_{n-1}(x)\\
&=(xF_n(x)-F_{n-1}(x))^2=(xF_n(x)-F_{n-1}(x))(x(xF_{n-1}(x)+F_{n-2}(x))-F_{n-1}(x))\\
&=xF_n(x)(x^2-1)F_{n-1}(x)+x^2F_n(x)F_{n-2}(x)-F_{n-1}(x)(x^2F_{n-1}(x)+xF_{n-2}(x)-F_{n-1}(x))\\
&=(x^2-1)xF_n(x)F_{n-1}(x)+x^2F_n(x)F_{n-2}(x)-x^2F_{n-1}^2(x)+F_{n-1}(x)F_{n-3}(x)\\
&=(x^2-1)xF_n(x)F_{n-1}(x)+x^2(xF_{n-1}(x)+F_{n-2}(x))F_{n-2}(x)-x^2F_{n-1}^2(x)+F_{n-1}(x)F_{n-3}(x)\\
&=(x^2-1)xF_n(x)F_{n-1}(x)+x^3F_{n-1}(x)F_{n-2}(x)+x^2F_{n-2}^2(x)-x^2F_{n-1}^2(x)+F_{n-1}(x)F_{n-3}(x)\\
&=(x^2-1)xF_n(x)F_{n-1}(x)+x^2F_{n-2}^2(x)+x^2F_{n-1}(x)(xF_{n-2}(x)-F_{n-1}(x))+F_{n-1}(x)F_{n-3}(x)\\
&=(x^2-1)xF_n(x)F_{n-1}(x)+x^2F_{n-2}^2(x)-x^2F_{n-1}(x)F_{n-3}(x)+F_{n-1}(x)F_{n-3}(x)\\
&=(x^2-1)xF_n(x)F_{n-1}(x)+x^2F_{n-2}^2(x)-(x^2-1)F_{n-1}(x)F_{n-3}(x)\\
&=x^2F_{n-2}^2(x)+(x^2-1)F_{n-1}(x)(xF_n(x)-F_{n-3}(x)).
\end{aligned}$$

Also solved by Brian D. Beasley, Scott H. Brown, Paul S. Bruckman, José Luis Díaz, L. A. G. Dresel, Walther Janous, Harris Kwong, Jaroslav Seibert, H.-J. Seiffert, Pantelimon Stănică, and the proposer.

Some Product

B-909 ***Proposed by J. Cigler, Universität Wien, Austria (Vol. 38, no. 5, November 2000)***

Consider an arbitrary sequence of polynomials $p_k(x)$ of the form $p_k(x)=x^{a_k}(x-1)^{b_k}$, where a_k and b_k are integers satisfying $a_k+b_k=k$ and $a_k\geq b_k+1\geq 0$. Let $L_{n,k}$ be the uniquely determined numbers such that $x^n=\Sigma L_{n,k}p_k(x)$. Show that

$$F_n=\Sigma L_{n,k}F_{a_k-b_k},$$

where F_n are the Fibonacci numbers.

If all $a_k-b_k\in\{1,2\}$, then we have $F_n=\Sigma L_{n,k}$. This generalized Proposition 2.2 of the paper "Fibonacci and Lucas Numbers as Cumulative Connection Constants" in *The Fibonacci Quarterly* **38.2** (2000):157-64.

Solution by L. A. G. Dresel, Reading, England

Let $a(k) = a_k$ and $b(k) = b_k$. Then by definition, $p_k(\alpha) = \alpha^{a(k)}(\alpha-1)^{b(k)}$. Since $\alpha(\alpha-1)=1$, we have $p_k(\alpha) = \alpha^{a(k)-b(k)}$ and, similarly, $p_k(\beta) = \beta^{a(k)-b(k)}$. Furthermore, from the definition of $L_{n,k}$, we have $\alpha^n = \Sigma L_{n,k} p_k(\alpha) = \Sigma L_{n,k}(\alpha^{a(k)-b(k)})$ and, similarly, $\beta^n = \Sigma L_{n,k}(\beta^{a(k)-b(k)})$. Hence,

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} = \Sigma L_{n,k}(\alpha^{a(k)-b(k)} - \beta^{a(k)-b(k)})/\sqrt{5} = \Sigma L_{n,k} F_{a(k)-b(k)}.$$

Also solved by Paul S. Bruckman, Reiner Martin, H.-J. Seiffert, and the proposer.

A Diophantine Equation

B-910 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France (Vol. 38, no. 5, November 2000)***

Solve the equation $p^n + 1 = \frac{k(k+1)}{2}$, where p is a prime number and k is a positive integer.

Remark: The case $p=2$ is Problem B-875 (*The Fibonacci Quarterly*, May 1999; see February 2000 for the solution).

Solution by Pantelimon Stănică & Charles White (jointly), Auburn University Montgomery, Montgomery, AL

Assume $n=0$. The equation transforms into $\frac{k(k+1)}{2} = 2$, which has no solution in integers. Thus, $n \geq 1$. Now, we write $(*)$ as $p^n = \frac{(k-1)(k+2)}{2}$. First, assume that k is odd, $k = 2s+1$, $s \geq 0$. We get $p^n = s(2s+3)$, and since p is prime, $s = p^\alpha$ and $2s+3 = p^\beta$, $0 \leq \alpha < \beta$. If $\alpha = 0$, then $s=1$, so $p^n = 2s+3 = 5$, which produces the solution $(p,k,n) = (5,3,1)$. Now, take $\alpha \geq 1$. Thus, $2p^\alpha + 3 = p^\beta$, with $1 \leq \alpha < \beta$, so p divides 3, which implies $p=3$ and $2\cdot 3^\alpha = 3\cdot(3^{\beta-1}-1)$. Therefore, $\alpha = 1$, $\beta = 2$, and we get $(p,k,n) = (3,7,3)$.

Now, assume k is even, $k = 2s$, $s \geq 1$. We get $p^n = (s+1)(2s-1)$. It follows that $s+1 = p^\alpha$ and $2s-1 = p^\beta$, with $\alpha > \beta$ or $0 \leq \alpha \leq \beta$. If $\alpha > \beta$, then $s=1$ and we get the solution $(p,k,n) = (2,2,1)$. If $\alpha = 0$, then $s=0$, which implies $p^n + 1 = 0$, obviously a contradiction. Assume that $1 \leq \alpha \leq \beta$, then $2p^\alpha - 3 = p^\beta$, so as before $p=3$ and we get $(p,k,n) = (3,4,2)$. Thus, the given equation has four solutions (p,k,n); namely, $(2,2,1)$, $(3,4,2)$, $(3,7,3)$, and $(5,3,1)$.

Also solved by Paul S. Bruckman, Charles K. Cook, L. A. G. Dresel, Walther Janous, Harris Kwong, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

Note: *The Elementary Problems Editor, Dr. Russ Euler, is in need of more easy yet elegant and nonroutine problems.*

❖❖❖