THE LEAST NUMBER HAVING 331 REPRESENTATIONS AS A SUM OF DISTINCT FIBONACCI NUMBERS

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(Submitted October 1999-Final Revision February 2000)

1. INTRODUCTION

Let A_n be the least number having exactly n representations as a sum of distinct Fibonacci numbers. Let $R(N)$ denote the number of representations of N as sums of distinct Fibonacci numbers, and let Zeck N denote the Zeckendorf representation of N, which is the unique representation of N as a sum of distinct, nonconsecutive Fibonacci Numbers. The sequence $\{A_n\}$ is sequence A013583 studied earlier [7], [9], where we list the first 330 terms; here, we extend our computer results by pencil and logic to calculate A_{331} and other "missing values." We list some pertinent background information.

Theorem 1: The least integer having F_k representations is $(F_k)^2-1$, and F_k is the largest value for $R(N)$ for N in the interval $F_{2k-2} \le N < F_{2k-1}$.

Theorem 2: Let N be an integer written in Zeckendorf form; if $N = F_{n+k}+K$, $F_n \le K < F_{n+1}$, we can write $R(N)$ by using the appropriate formula:

$$R(N) = R(F_{n+2p}+K) = pR(K)+R(F_{n+1}-k-2),\ k=2p; \tag{1.1}$$

$$R(N) = R(F_{n+2p+1}+K) = (p+1)R(K),\ k=2p+1; \tag{1.2}$$

$$R(N) = R(N-F_{2w})+R(F_{2w+1}-N-2),\ F_{2w} \le N < F_{2w+1}. \tag{1.3}$$

Theorem 3: Zeck A_n ends in $\ldots+F_{2c}$, $c \ge 2$. If Zeck N ends in $\ldots+F_{2c+2k+1}+F_{2c}$, $c \ge 2$, then

$$R(N) = R(N-1)R(F_{2c}) = cR(N-1).$$

If Zeck $A_n = F_m + K$, then $F_m < A_n < F_m + F_{m-2}$.

Lemma 1: If $\{b_n\}$ is a sequence of natural numbers such that $b_{n+2} = b_{n+1}+b_n$, then

$$R(b_n-1) = R(b_{n+1}-1) = k$$

for all sufficiently large n (see [8]).

Lemma 1 and Theorem 3 are useful in calculating A_n when n is composite. Theorems 2 and 3 are proved in [2] and [3], while Theorem 1 is the main result of [1].

Theorem 4: If $n = R(A_n)$ is a prime, then Zeck A_n ;is the sum of even-subscripted Fibonacci numbers only. If Zeck A_n begins with F_{2k+1}, then $n = R(A_n)$ cannot be prime.

Proof: Theorem 3 and (1.2) show that a change in parity in subscripts indicates that at least one pair of factors exists for $R(N)$.

Lemma 2: For integers N such that $F_{2k} \le A_n \le N < F_{2k+1}$ and $R(N) = n$,

$$F_{k-1} \le R(A_n - F_{2k}) \le R(F_{2k+1} - N - 2) \le R(F_{2k+1} - A_n - 2) \le F_k. \tag{1.4}$$

Proof: The pair of exterior endpoints are a consequence of Theorem 1. The pair of interior endpoints reflect symmetry about the center of the interval, since $R(N)$ is a palindromic sequence within each such interval $F_{2k} \le N < F_{2k+1} - 2$.

If Zeck $n = F_k + K$, $0 < K \le F_{k-2}$, then $A_n > F_{2k-1}$, where we note that we are relating the Zeckendorf representations of A_n and of $R(A_n)$. In our extensive tables, Zeck A_n begins with F_{2k-1}, F_{2k}, F_{2k+1}, or F_{2k+2}, while all values for n, $1 \le n = R(N) \le F_k$, appear for $N < F_{2k+1}$, but this has not been proved. The first 330 values for A_n are listed in [7], too long a table to repeat here. Our computer results conclude with $A_{466} = 229971$; there are 69 "missing values" for n between 330 and 466. We also have complete tables for $R(N)$ for all $N < F_{22}$, not included here, which shorten the work but are not essential to follow the logic in solving for A_n given n.

2. THE CALCULATION OF A_{331}

Since A_n is known for all $n \le 330$ and, for all n such that $A_n < F_{28}$, and since 331 is prime, we can find A_{331} by listing successive addends for Zeck A_n, and choosing the smallest possibility at each step. Let $N = F_{28} + K^*$, for $F_{28-2q} \le K < F_{29-2q}$. Then

$$R(N) = qR(K) + R(F_{29-2q} - K - 2) \tag{2.1}$$

by (1.1), and the maximum possible value for $R(N)$ is

$$\max R(N) = qF_{15-q} + F_{14-q}$$

by Theorem 1. Since $F_{2k} \le A_n < F_{2k} + F_{2k-2}$, $2 \le q$. We summarize in Table 1.

TABLE 1

$N = F_{28} + K,\ F_{28-2q} \le K < F_{29-2q}$

	$\max R(N)$: $qF_{15-q} + F_{14-q}$
$q = 2$:	$2F_{13} + F_{12} = 466 + 144 = 610$
$q = 3$:	$3F_{12} + F_{11} = 432 + 89 = 521$
$q = 4$:	$4F_{11} + F_{10} = 356 + 55 = 411$
$q = 5$:	$5F_{10} + F_9 = 275 + 34 = 309$

Notice that maximum values for $R(N)$ for $q \ge 5$ are smaller than 331. For our purposes, the smallest possibility is $q = 4$, or $N = F_{28} + F_{20} + K$. We write Table 2 to determine the third possible even subscript in Zeck N when $q = 4$.

Start with $w = 3$ in Table 2, the smallest possibility, with $F_{14} \le K < F_{15}$. Solve the Diophantine equation $14A + 5B = 331$, $13 < A \le 21$, which has $14(19) + 5(13) = 331$. By Lemma 2, since $A_{19} = F_{14} + A_7$, $7 \le B = R(F_{15} - K - 2) \le 19 - 7 = 12$. Thus, $B \ne 13$ and $w \ne 3$.

TABLE 2

$$N = F_{28} + F_{20} + K,\ F_{20-2w} \le K < F_{21-2w}$$
$$R(N) = (5w-1)\,R(K) + 5R(F_{21-2w} - K - 2)$$
$$\max R(N):\ (5w-1)F_{11-w} + 5F_{10-w}$$

$w = 1$:	$4F_{10} + 5F_9 = 220 + 170 = 390$
$w = 2$:	$9F_9 + 5F_8 = 306 + 105 = 411$
$w = 3$:	$14F_8 + 5F_7 = 294 + 65 = 359$
$w = 4$:	$19F_7 + 5F_6 = 247 + 40 = 287$

Next take $w = 2$ in Table 2, with $F_{16} \le K < F_{17}$, and solve $9A + 5B = 331$, $21 < A \le 34$, which has $9(34) + 5(5) = 331$, but $A_{34} = F_{16} + A_{13}$, so that we must have $13 \le B \le 21$; $B = 5$ is too small. We also find $9(29) + 5(14) = 331$, which is plausible since $A_{29} = 1050 = F_{16} + A_8$, and $8 \le B = 14 \le 21$. However, this combination of values does not appear in the computer printouts; only $N = 1050$, 1152, 1189 have $R(N) = 29$ for $N < F_{16} + F_{14}$, so $B = 8$, 21, 18, 11, 17, or 12, but not 14. However, we can verify that $B \ne 14$ either by assuming that the next term is F_{14} and calculating one more step, or by noting that we are solving $A = R(K) = 29$ for some K which also has $R(F_{17} - K - 2) = 14$ and $R(K - F_{16}) = 29 - 14 = 15$. We must have $K - F_{16} \ge A_{15} = F_{13} + F_8 + F_4$ or $K = F_{16} + F_{14} + K'$. Then, because $F_{17} - K - 2 = F_{13} - K' - 2 < A_{14} = F_{13} + 16$, we cannot have $R(F_{17} - K - 2) = 14 = B$, a contradiction. The last viable solution $9(24) + 5(23) = 331$ has B too large. Thus, $w \ne 2$.

Finally, take $w = 1$, with $F_{18} \le K < F_{19}$. Solve $4A + 5B = 331$ for $34 < A \le 55$, obtaining $4(49) + 5(27) = 331$ and $4(44) + 5(31) = 331$, where $4(39) + 5(35) = 331$ has B too large. From the computer printout, $A_{44} = F_{18} + A_{12} = 2744$, but $R(F_{19} - 2744 - 2) = 32$, not 31. The next occurrence of $R(K) = 44$ in our computer table is for $K = 2791$ for which $31 = R(F_{19} - 2791 - 2)$; and since 2791 is the smallest integer that satisfies all of the parameters, we have a solution. Without such a table, one could assume that F_{18} is the next term, and compute the term following F_{18}. We now have

$$A_{331} = F_{28} + F_{20} + 2791 = 327367.$$

Let us make use of our work thus far. In Table 2, $w = 3$ has $14F_8 + 5F_7 = 359$, one of the "missing values." Since we cannot write a smaller solution,

$$A_{359} = F_{28} + F_{20} + A_{21} = 317811 + 6765 + 440 = 325016.$$

Also, Table 1, $q = 4$, $N = F_{28} + (F_{20} + K)$ has $R(N) = 359$ for $4(76) + 55 = 359$, or for $N = F_{28} + A_{76} = 317811 + 7205$, which gives the same result.

3. THE CALCULATION OF A_{339}

The second missing value on our list is 339. We can find A_{339} with very little effort, although $339 = 3 \cdot 113$ is not a prime. Since $A_{113} = F_{24} + K$, $N = F_{28} + F_{23} + \cdots$ has $R(N) = 3R(F_{23} + \cdots)$, and A_{113} is too large to appear as the second factor. Now, taking $q = 4$, for $N = F_{28} + (F_{20} + K)$, $4(74) + 43 = 339$, and $A_{74} = 8187$ while $R(F_{21} - 8187 - 2) = R(2757) = 43$; in fact, $2757 = A_{43}$. Then

$$N = A_{339} = F_{28} + A_{74} = 317811 + 8187 = 325998.$$

We have also generated

$$N = F_{28} + A_{89} = 317811 + 7920 = 325731$$

which has $R(N) = 411$ from Table 1, $q = 4$, while Table 2, $w = 2$, gives $A_{411} = F_{28} + F_{20} + A_{34}$ which is the same result. Just as for 339, while we can factor $411 = 3 \cdot 137$, A_{137} is too large. Again from Table 2, $w = 2$, changing A and B slightly, we find $9F_9 + 5F_7 = 371$, also on our list. If we take $K = 1427 = F_{16} + F_{14} + F_{10} + F_6$, then $R(K) = 34$, $R(F_{21} - K - 2) = 13$; the only other value for K in this interval such that $R(K) = 34$ is A_{34} but $R(F_{21} - A_{34} - 2) = 21$, so 1427 is the smallest we can take for K. Thus, we write

$$A_{371} = F_{28} + F_{20} + 1427 = 326003.$$

We next illustrate how to use factoring to find A_n when n is composite, using Lemma 1 and Theorem 3. Let $n = 410 = 41 \cdot 10$:

$$\begin{aligned}
&A_{10} = 105 = F_{11} + F_7 + F_4,\\
&A_{41} = 2736 = F_{18} + F_{12} + F_6,\\
&41 = R(F_{18} + F_{12} + F_6 + F_1 - 1) = R(F_{28} + F_{22} + F_{16} + F_{11} - 1),\\
&41 \cdot 10 = R(F_{28} + F_{22} + F_{16} + A_{10}).
\end{aligned}$$

$N = F_{28} + F_{22} + F_{16} + F_{11} + F_7 + F_4 = 336614$ has $R(N) = 410$. Writing $R(N)$ as $205 \cdot 2$ gives the same solution, while $82 \cdot 5$ gives a slightly larger solution. $N = A_{410}$ if there is no smaller solution using the even subscript formula. We can easily see that $N \neq F_{28} + F_{20} + K$ from our earlier work, so we test out $N = F_{28} + F_{22} + K$ in Table 3.

TABLE 3

$$N = F_{28} + F_{22} + K,\ F_{22-2p} \le K < F_{23-2p}$$

$$R(N) = (4p-1)R(K) + 4R(F_{23-2p} - K - 2)$$

$$\max R(N):\ (4p-1)F_{12-p} + 4F_{11-p}$$

$p = 1$:	$3F_{11} + 4F_{10} = 267 + 220 = 487$
$p = 2$:	$7F_{10} + 4F_9 = 385 + 136 = 521$
$p = 3$:	$11F_9 + 4F_8 = 374 + 84 = 458$
$p = 4$:	$15F_8 + 4F_7 = 315 + 52 = 367$
$p = 5$:	$19F_7 + 4F_6 = 247 + 24 = 271$

The smallest choice to generate 410 is $p = 3$ for $F_{16} \le K < F_{17}$ which requires that we solve $11A + 4B = 410$ for $A \le 34$ which, in turn, gives us $11(30) + 4(20) = 410$; $A_{30} = 1092 = F_{16} + 105$ and $R(F_{17} - A_{30} - 2) = 20$, so that

$$A_{410} = F_{28} + F_{22} + F_{16} + 105,$$

the same result as by factoring.

Note that Table 3 provides more "missing values" on our list. Here, $p = 4$ gives $R(N) = 367$ for $N = F_{28} + F_{22} + A_{21}$, which easily demonstrates that $N \neq F_{28} + F_{20} + K$, so $A_{367} = 335962$, the same result as $A_{367} = F_{28} + A_{97}$ by working with Table 1, $q = 3$. Furthermore,

$$A_{458} = F_{28} + A_{123} = 317811 + 18866 = 336677$$

comes from $q = 3$ of Table 1, and $A_{458} = F_{28} + F_{22} + A_{34}$ comes from $p = 3$ above.

We expect to see all "missing values" $n < F_{14} = 377$ appearing for $N = F_{28} + K$ based on our previous experience, but we have been unable to prove that all $n = R(N)$, $1 \le n \le F_k$, will appear for $N = F_{2k} + K$. Generating some of them will take patience, especially for a value such as $n = 421$ which has no solution for $A_n = F_{28} + K$. One can generate more tables such as Table 4 similarly to Tables 1 through 3, or one can list possible successive subscripts for Zeck A_n and evaluate each case.

Some results, verifiable in other ways, can be read from the tables. From Table 4. below, we have

$$A_{610} = F_{28} + F_{24} + A_{89} \quad \text{and} \quad A_{542} = F_{28} + F_{24} + A_{55}.$$

However, $A_{555} = A_{610} + 5 < F_{28} + F_{24} + A_{144}$. Table 1 gives

$$A_{610} = F_{28} + A_{233} \text{ (the same result)} \quad \text{and} \quad A_{521} = F_{28} + A_{144}.$$

Table 3 gives

$$A_{487} = F_{28} + F_{22} + A_{89} \quad \text{and} \quad A_{521} = F_{28} + F_{22} + A_{55} \text{ (the same result).}$$

Table 4 gives $R(N) = 333$ for $N = F_{28} + F_{24} + A_{21}$, where Zeck N uses only even-subscripted Fibonacci numbers, but $A_{333} = 209668 < N$. One must verify that N is the smallest possible, especially if $R(N)$ is composite.

TABLE 4

$$N = F_{28} + F_{24} + K, \ F_{24-2p} \le K < F_{25-2p}$$

$$R(N) = (3p-1)R(K) + 3R(F_{25-2p} - K - 2)$$

$$\max R(N):\ (3p-1)F_{13-p} + 3F_{12-p}$$

$p = 1$:	$2F_{12} + 3F_{11} = 288 + 267 = 555$
$p = 2$:	$5F_{11} + 3F_{10} = 445 + 165 = 610$
$p = 3$:	$8F_{10} + 3F_9 = 440 + 102 = 542$
$p = 4$:	$11F_9 + 43F_8 = 374 + 63 = 437$
$p = 5$:	$14F_8 + 3F_7 = 294 + 39 = 333$

By constructing N taking one even-subscripted Fibonacci number at a time, one can find A_n for n prime, $n < 466$; some solutions are very short, while others take patience. Prime values for n in Table 5 can be found for $N = F_{28} + K$ except for $n = 421$, 439, and 461, which need $N = F_{30} + K$. The composites n for which $A_n > F_{28} + K$, found by considering factors of n, need $N = F_{29} + K$. Note that only the subscripts in Zeck A_n are listed in Table 5.

The calculations of A_n for n prime and of A_n, where Zeck A_n has even subscripts only agree with D. Englund [4], [5], and with computations using "Microsoft Excel" by M. Johnson. Of the composites $n = R(A_n)$, where A_n contains an odd-subscripted term, there are very many cases to consider and thus checking is more difficult. Each composite n starred in the table can be computed from its factors and has $A_n < N$, where $R(N) = n$ and Zeck N contains even-subscripted Fibonacci numbers only.

TABLE 5. "Missing Values" for n, $331 \le n = R(N) \le 465$

n prime

n	A_n	Zeck A_n
331	327367	28,20,18,12,10,6
347	336067	28,22,14,12,8,4
349	339528	28,22,18,16,14,10,4
353	338185	28,22,18,10,8,4
359	325016	28,20,14,10,6
367	335962	28,22,14,10,6
373	336588	28,22,16,10,8,4
379	338690	28,22,18,14,12,10,6
383	338638	28,22,18,14,12,6,4
389	336944	28,22,16,14,10,4
397	342688	28,22,20,14,8,4
401	338648	28,22,18,14,12,8
409	343476	28,22,20,16,12,10,4
419	338656	28,22,18,14,12,8,6
421	839994	30,20,16,12,10,4
431	343714	28,22,20,16,14,10,8
433	343426	28,22,20,16,12,6
439	841557	30,20,18,12,8,4
443	343447	28,22,20,16,12,8,6
449	367292	28,24,18,14,12,6
457	367923	28,24,18,16,12,8,6
461	851181	30,22,16,14,10,6,4
463	338562	28,22,18,14,10,8,4

n composite

n	A_n	Zeck A_n
339	325998	28,20,16,14,10,4
371	326003	28,20,16,14,10,6
381	339533	28,22,18,16,14,10,6
391	336674	28,22,16,12,8
394	343709	28,22,20,16,14,10,4
396*	337224	28,22,17,11,7,4
402*	336690	28,22,16,12,9,4
404*	343722	28,22,20,16,14,10,7,4
406	336661	28,22,16,12,6
407	338258	28,22,18,12,6
410*	336614	28,22,16,11,7,4
411	325731	28,20,16,12,8,4
412*	365326	28,24,16,12,7,4
413	336716	28,22,16,12,10,6
415	339300	28,22,18,16,12,10,6
417	336682	28,22,16,12,8,6
422	371960	28,24,20,16,8,6
423*	338580	28,22,18,14,11,6
425	338279	28,22,18,12,8,6
426	336949	28,22,16,14,10,6
427	372015	28,24,20,16,10,8,6
428*	372468	28,24,20,16,14,12,7,4
429*	337287	28,22,17,12,8,4
430	338635	28,22,18,14,12,6
434	339156	28,22,18,16,10,6
435*	338363	28,22,18,13,8,4
436*	338266	28,22,18,12,7,4
437	343337	28,22,20,16,10,6
438	338512	28,22,18,14,8,6
444*	339253	28,22,18,16,12,7,4
446	367957	28,24,18,16,12,10,6
447*	530063	29,21,19,15,11,6
448*	338643	28,22,18,14,12,7,4
450*	338829	28,22,18,15,11,8,4
451*	544635	29,23,17,12,6
452*	527110	29,21,17,13,11,7,4
453	371350	28,24,20,14,8,6
454*	526877	29,21,17,11,7,4
455*	340426	28,22,19,15,11,8,4
456*	338520	28,22,18,14,9,4
458	336677	28,22,16,12,8,4
459*	544580	29,23,17,11,6
460*	343434	28,22,20,16,12,7,4
462*	337389	28,22,17,13,9,4
464*	338376	28,22,18,13,9,4
465	338274	28,22,18,12,8,4

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AMS Classification Numbers: 11B39, 11B37, 11Y55

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