A NOTE ON ORTHOGONAL POLYNOMIALS

J.C. Ahuja and S.W. Nash
University of Alberta, Calgary, Canada

1. INTRODUCTION

The recurrence relation for orthogonal polynomials $q_n(x)$ (leading coefficient one) associated with the density function $f(x)$ over the interval $[a, b]$ is derived explicitly in terms of the moments of $f(x)$. Further, an alternative proof is given of the theorem that if $f(x)$ is symmetrical about $x = 0$, then the polynomials $q_n(x)$ are even or odd functions according as n is even or odd.

2. RESULTS

Let $f(x)$ denote the density of a distribution function $F(x)$ with infinitely many points of increase in the finite or infinite interval a, b, and let the moments

$$m_r = \int_a^b x^r f(x)dx$$

exist for $r = 0, 1, 2, \ldots$.

It is well known, see Szegö[1], that there exists a sequence of polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$, ... uniquely determined by the following conditions:

(a) $p_n(x)$ is a polynomial of precise degree n in which the coefficient of x^n is positive.

(b) the system $p_n(x)$ is orthonormal, that is

$$\int_a^b p_m(x)p_n(x)f(x)dx = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

If, on the other hand, $F(x)$ has only N points of increase, then the $p_n(x)$ exist and are uniquely determined for $n = 0, 1, \ldots, N-1$; and if $F(x)$ has only finitely many finite moments, say m_{2k} or

m_{2k+1} exist, then the $p_n(x)$ exist and are uniquely determined for $n = 0, 1, \ldots, k$.

The polynomials $p_n(x)$ satisfying conditions (a) and (b) above are of the form

$$p_n(x) = \frac{1}{\sqrt{D_{n-1} D_n}} \begin{vmatrix} m_0 & m_1 & m_2 \cdots\cdots & m_n \\ m_1 & m_2 & m_3 \cdots\cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} \cdots & m_{2n-1} \\ 1 & x & x^2 \cdots\cdots & x^n \end{vmatrix}, \tag{1}$$

where

$$D_n = \begin{vmatrix} m_0 & m_1 & m_2 \cdots\cdots & m_n \\ m_1 & m_2 & m_3 \cdots\cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} \cdots & m_{2n-1} \\ m_n & m_{n+1} & m_{n+2} \cdots & m_{2n} \end{vmatrix}$$

and where the leading coefficients of $p_n(x)$ are

$$\sqrt{\frac{D_{n-1}}{D_n}} .$$

If we now define the polynomials $q_n(x)$ as

$$q_n(x) = \sqrt{\frac{D_n}{D_{n-1}}}\; p_n(x) , \tag{2}$$

then the $q_n(x)$ are orthogonal polynomials whose leading coefficients are always one.

According to Szegö [1], the following relation holds for any three consecutive orthonormal polynomials:

(3) $$p_n(x) = (A_n x + B_n)p_{n-1}(x) - C_n p_{n-2}(x), \quad n = 2, 3, \ldots$$

where A_n, B_n and C_n are constants, $A_n > 0$ and $C_n > 0$. If the highest coefficient of $p_n(x)$ is denoted by k_n, then

$$A_n = \frac{k_n}{k_{n-1}}, \quad C_n = \frac{A_n}{A_{n-1}} = \frac{k_n k_{n-2}}{k_{n-1}^2}$$

Since $k_n = \sqrt{\frac{D_{n-1}}{D_n}}$, we shall have

$$A_n = \sqrt{\frac{D_{n-1}}{D_n D_{n-2}}}, \quad C_n = \sqrt{\frac{D_{n-3}}{D_n}} \left(\frac{D_{n-1}}{D_{n-2}}\right)^{3/2}$$

The relation (3) then becomes

(4) $$p_n(x) = \left(\frac{D_{n-1}}{\sqrt{D_n D_{n-2}}}\, x + B_n\right) p_{n-1}(x) - \sqrt{\frac{D_{n-3}}{D_n}} \left(\frac{D_{n-1}}{D_{n-2}}\right)^{3/2} p_{n-2}(x)$$

Multiplying both sides of (4) by

$$\sqrt{\frac{D_n}{D_{n-1}}}$$

and using (2), we get

(5) $$q_n(x) = \left(x + \frac{\sqrt{D_n D_{n-2}}}{D_{n-1}}\, B_n\right) q_{n-1}(x) - \frac{D_{n-1} D_{n-3}}{D_{n-2}^2}\, q_{n-2}(x)$$

To find B_n, let us suppose that k'_n is the coefficient of x^{n-1} in $p_n(x)$, while k_n is the coefficient of x^n in $p_n(x)$. By equating the coefficients of x^{n-1} on both sides of (3), we get

$$k'_n = A_n k'_{n-1} + B_n k_{n-1}$$

which gives

(6) $$B_n = \frac{k'_n}{k_{n-1}} - A_n \frac{k'_{n-1}}{k_{n-1}}$$

But

$$A_n = \frac{k_n}{k_{n-1}} ,$$

so that (6) can be written as

$$B_n = \frac{k_n}{k_{n-1}} \left[\frac{k'_n}{k_n} - \frac{k'_{n-1}}{k_{n-1}} \right] \tag{7}$$

Let D^*_n denote the determinant obtained by deleting the (n+1)th row and the nth column of D_n. Then

$$k'_n = - \frac{D^*_n}{\sqrt{D_n D_{n-1}}}$$

Substituting for k_n, k'_n and k'_{n-1} in (7), we get

$$B_n = \frac{D_{n-1}}{\sqrt{D_n D_{n-2}}} \left[- \frac{D^*_n}{D_{n-1}} + \frac{D^*_{n-1}}{D_{n-2}} \right] \tag{8}$$

Using the value of B_n given by (8) in (5), we obtain

$$q_n(x) = \left(x - \frac{D^*_n}{D_{n-1}} + \frac{D^*_{n-1}}{D_{n-2}} \right) q_{n-1}(x) - \frac{D_{n-1} D_{n-3}}{D^2_{n-2}} q_{n-2}(x) . \tag{9}$$

Thus (9) gives the recurrence relation for orthogonal polynomials associated with the density function f(x) explicitly in terms of the moments of f(x). The recurrence relation (9) is valid also for n = 1 if we set $D^*_0 = 0$, $D_{-1} = 1$ and $D_{-2} = 0$.

If the density function f(x) is symmetrical about x = 0, that is, if f(-x) = f(x) and a = -b, then

$$m_1 = m_3 = \ldots\ldots = m_{2r+1} = \ldots\ldots = 0$$

If the odd order moments are all zero, we shall prove below in Theorem 1 that D^*_n vanishes for n = 1, 2, ... which will imply that $B_n = 0$ for n = 1, 2, ...

We shall also prove below in Theorem 2 that, in this case, the polynomials $q_n(x)$ are even or odd function according as n is even or odd.

The recurrence relation for orthogonal polynomials associated with the symmetrical density function $f(x)$ is then obtained as

$$q_n(x) = xq_{n-1}(x) - \frac{D_{n-1}D_{n-3}}{D_{n-2}^2} q_{n-2}(x) \tag{10}$$

In particular, for $n = 0, 1, 2, 3,$ and 4, the orthogonal polynomials associated with the symmetrical density function $f(x)$ are obtained as follows:

$$q_0(x) = 1$$

$$q_1(x) = x$$

$$q_2(x) = x^2 - m_2$$

$$q_3(x) = x^3 - \frac{m_4}{m_2} x$$

$$q_4(x) = x^4 - \frac{m_6 - m_2 m_4}{m_4 - m_2^2} x^2 + \frac{m_2 m_6 - m_4^2}{m_4 - m_2^2}$$

We now prove the following two Theorems:

Theorem 1. Let $D = \left[d_{ij} \right]$ be an $(n \times n)$ matrix where $d_{ij} = 0$ for $i+j$ odd, and d_{ij} is arbitrary for $i+j$ even. Let D^* be an $(n-1)\times(n-1)$ matrix obtained by deleting the uth row and the vth column of D such that $u+v$ is odd. Then the determinant of D^* is zero.

Proof. To prove the theorem we consider two cases: (1) n even and (2) n odd.

Case 1 $n = 2k$ (even)

Let us assume that we get D^* by deleting an odd row and an even column. Then by shifting rows and columns of D^*, we obtain

$$D^{**} = \begin{bmatrix} A_1 & A_2 \\ & \\ A_3 & A_4 \end{bmatrix}$$

where D^{**} is a matrix of $(2k-1)x(2k-1)$ elements and

A_1 = k x k matrix with zero elements

A_2 = k x (k-1) matrix with arbitrary elements

A_3 = (k-1) x k matrix with arbitrary elements

A_4 = (k-1) x (k-1) matrix with zero elements.

If we now take the Laplace expansion of D^{**} by (k x k) minors, then it can be easily seen that the determinant of D^{**} is zero, which will imply that the determinant of D^* is zero.

The result also follows if we take D^* by deleting an even row and an odd column.

Case 2 n = 2k+1 (odd)

In this case, we obtain

$$D^{**} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where D^{**} is a matrix of (2k x 2k) elements and

B_1 = k x (k+1) matrix with zero elements

B_2 = k x (k-1) matrix with arbitrary elements

B_3 = k x (k+1) matrix with arbitrary elements

B_4 = k x (k-1) matrix with zero elements.

If we take the Laplace expansion of D^{**} by (k x k) minors, then we shall have the determinant of D^{**} equal to zero, which will imply that the determinant of D^* is zero.

Theorem 2 Let $q_n(x)$, defined by (2), be the orthogonal polynomials associated with the density function $f(x)$ symmetrical about $x = 0$. Then the polynomials $q_n(x)$ are even or odd functions according as n is even or odd.

Proof If the density function $f(x)$ is symmetrical about $x = 0$, then all the odd order moments are zero, that is

$$m_1 = m_3 = \ldots = m_{2r+1} = \ldots = 0$$

The proof of the theorem follows immediately by expanding $p_n(x)$, defined by (1), in terms of the last row of the determinant and making use of the result of Theorem 1.

REFERENCE

1. Szegö, G., Orthogonal Polynomials, rev. ed., Amer. Math. Soc. Colloquium Publications, Vol. 23, New York, 1959.

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