SOME ORTHOGONAL POLYNOMIALS RELATED TO FIBONACCI NUMBERS

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1. We consider polynomials $f_n(x)$ such that

(1) $$f_{n+2}(x) = (x+2n+p+1)f_{n+1}(x) - (n^2+pn+q)f_n(x) \qquad (n = 0, 1, 2, \ldots),$$

where

(2) $$f_0(x) = 0, \; f_1(x) = 1 .$$

It follows at once that $f_n(x)$ is a polynomial in x of degree $n-1$ for $n \geq 1$. The parameters p, q are arbitrary but we shall assume that

(3) $$p^2 - 4q \neq 0 .$$

Let α, β denote the roots of the equation

(4) $$x^2 - px + q = 0 .$$

In view of (3), the roots α, β are distinct and

(5) $$\alpha+\beta = p, \quad \alpha\beta = q .$$

We shall construct a generating function for $f_n(x)$:

(6) $$F(t) = F(t, x) = \sum_{n=0}^{\infty} f_n(x)t^n/n!$$

It is easily verified that (1), (2) and (6) imply

(7) $$(1-t)^2F''(t) - [(x+(p+1)(1-t)]F'(t) + qF(t) = 0 ,$$

where the primes indicate differentiation with respect to t.

It is convenient to define an operator.

(8) $$\Delta = (1-t)^2D^2 - (p+1)D + q \quad (D = d/dt) .$$

Thus (7) becomes

(9) $$\Delta F(t) = xF'(t) .$$

Supported in part by NSF grant GP 1593.

Consider

$$\Delta(1-t)^{-\alpha-k} = \{(\alpha+k)(\alpha+k+1)-(p+1)(\alpha+k)+q\}(1-t)^{-\alpha-k} .$$

Making use of (4) we find that this reduces to

$$\Delta(1-t)^{-\alpha-k} = k(2\alpha-p+k)(1-t)^{-\alpha-k} . \tag{10}$$

Thus, if we put

$$\Phi(t,\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_k x^k}{k!(2\alpha-p+1)_k}(1-t)^{-\alpha-k} , \tag{11}$$

where

$$(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1) ,$$

we get

$$\Delta\Phi(t,\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_{k+1} x^{k+1}}{k!(2\alpha-p+1)_k}(1-t)^{-\alpha-k-1} .$$

We have therefore

$$\Delta\Phi(t,\alpha) = x\Phi'(t,\alpha) \tag{12}$$

and in exactly the same way

$$\Delta\Phi(t,\beta) = x\Phi'(t,\beta) . \tag{13}$$

It follows from (11) that

$$\Phi(t,\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_k x^k}{k!(2\alpha-p+1)_k} \sum_{n=0}^{\infty} \frac{(\alpha+k)_n}{n!} t^n$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(\alpha)_{n+k} x^k}{k!(2\alpha-p+1)_k} .$$

If we put

$$\phi_n(x,\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha)_{n+k} x^k}{k!(2\alpha-p+1)_k} , \tag{14}$$

then we have

$$\Phi(t, \alpha) = \sum_{n=0}^{\infty} \phi_n(x, \alpha)t^n/n! \quad . \tag{15}$$

Note that (14) implies

$$\phi_n(x, \alpha) = (\alpha)_n \cdot {}_1F_1(\alpha+n; 2\alpha-p+1; x) \quad , \tag{16}$$

where ${}_1F_1$ denotes a hypergeometric function in the usual notation.

2. If we make use of (12) and (15) we find without much difficulty that $\phi_n(x, \alpha)$ satisfies the recurrence

$$\phi_{n+2}(x, \alpha) = (x+2n+p+1)\phi_{n+1}(x, \alpha) - (n^2+pn+q)\phi_n(x, \alpha) \quad (n \geq 0) \quad . \tag{17}$$

Clearly $\phi_n(x, \beta)$ satisfies the same recurrence. Thus any linear combination

$$\psi_n(x) = A\phi_n(x, \alpha) + B\phi_n(x, \beta) \quad ,$$

where A, B, are independent of n but may depend on x, α, β, will also satisfy (17).

We choose A, B so that

$$\psi_0(x) = 0, \quad \psi_1(x) = 1 \quad . \tag{18}$$

This requires

$$AC = \phi_0(x, \beta), \qquad BC = -\phi_0(x, \alpha) \quad ,$$

where

$$C = \phi_1(x, \alpha)\phi_0(x, \beta) - \phi_1(x, \beta)\phi_0(x, \alpha) \quad . \tag{19}$$

It is clear by comparison of (17) and (18) with (1) and (2) that

$$\psi_n(x) = f_n(x) \qquad (n = 0, 1, 2, \ldots) \quad .$$

We have therefore

$$f_n(x) = \frac{\phi_n(x, \alpha)\phi_0(x, \beta) - \phi_n(x, \beta)\phi_0(x, \alpha)}{C} \tag{20}$$

with C defined by (19).

Thus by (6) and (15)

$$F(t) = C^{-1}\left\{\Phi(t,\alpha)\phi_0(x,\beta) - \Phi(t,\beta)\phi_0(x,\alpha)\right\}, \tag{21}$$

so that we have obtained a generating function for $f_n(x)$.

3. In addition to the polynomial $f_n(x)$ we may construct a second solution $g_n(x)$ of (1) such that

$$g_0(x) = 1, \quad g_1(x) = x+p+1 \quad . \tag{22}$$

Thus $g_n(x)$ is a polynomial in x of degree n. By exactly the same method we have used above, we find that

$$g_n(x) = -2\,\frac{\phi_n(x,\alpha)\phi_1(x,\beta) - \phi_n(x,\beta)\phi_1(x,\alpha)}{C} + (x+p)f_n(x) \quad . \tag{23}$$

If we put

$$G(t) = G(t,x) = \sum_{n=0}^{\infty} g_n(x)\, t^n/n! \tag{24}$$

it follows that

$$G(t) = -2\,\frac{\Phi(t,\alpha)\phi_1(x,\beta) - \Phi(t,\beta)\phi_1(x,\alpha)}{C} + \frac{x+p}{C}\left(\Phi(t,\alpha)\phi_0(x,\beta) - \Phi(t,\beta)\phi_0(x,\alpha)\right) \quad . \tag{25}$$

If the coefficient n^2+pn+q occurring in (1) is positive for all $n \geq 0$ then by a known result [1] we can assert that the polynomials $g_n(x)$ are orthogonal on the real line with respect to some weight function. The same remark applies to the $f_n(x)$. It would be of interest to explicitly determine these weight functions.

4. We have assumed in the above discussion that α and β are distinct. When α and β are equal we may replace (1) by

$$f_{n+2}(x) = (x+2n+2\alpha+1)f_{n+1}(x) - (n+\alpha)^2 f_n(\alpha) \qquad (n = 0, 1, 2, \ldots) . \tag{26}$$

We now put

$$\phi_n(x) = \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{k!\,k!} \quad x^k \quad , \tag{27}$$

$$\Phi\,(t) = \sum_{n=0}^{\infty} \phi_n(x)t^n/n! = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{k!\,k!} \quad (1-t)^{-a-k} \quad . \tag{28}$$

It is easily verified that

$$\phi_{n+2}(x) = (x+2n+2a+1)\phi_{n+1}(x) - (n+a)^2\phi_n(x) \qquad (n = 0, 1, 2, \ldots) \tag{29}$$

and that

$$\Delta\Phi\,(t) = x\Phi'(t) \quad . \tag{30}$$

As a second solution of (26) we take

$$\psi_n(x) = \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{k!\,k!} \,(\sigma_{n+k}(a) - 2\sigma_k)\, x^k \quad , \tag{31}$$

where

$$\sigma_k(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+k-1} \quad , \tag{32}$$

$$\sigma_k = \sigma_k(1)\ 1\ 1 + \frac{1}{2} + \cdots + \frac{1}{k} \quad .$$

We omit the proof that $\psi_n(x)$ does indeed satisfy (26).

It is convenient to put

$$\Psi\,(t) = \sum_{n=0}^{\infty} \psi_n(x) \quad t^n/n! \quad . \tag{33}$$

It can be verified that $\Psi(t)$ also satisfies (30).

If we now put

$$f_n(x) = \frac{\phi_n(x)\psi_0(x) - \phi_0(x)\psi_n(x)}{\phi_1(x)\psi_0(x) - \phi_0(x)\psi_1(x)} \qquad (n = 0, 1, 2, \ldots) \quad , \tag{34}$$

then we have

(35) $$f_0(x) = 0, \quad f_1(x) = 1 \quad .$$

Thus $f_n(x)$ is a polynomial of degree $n-1$ in x for $n \geq 1$ and is the unique solution of (26) that satisfies (35).

Similarly if we put

(36) $$g_n(x) = 2\,\frac{\phi_1(x)\psi_n(x) - \psi_1(x)\phi_n(x)}{\phi_1(x)\psi_0(x) - \phi_0(x)\psi_0(x)} + (x+2\alpha+1)f_n(x)$$

then

(37) $$g_0(x) = 2, \quad g_1(x) = x + 2\alpha+1 \quad .$$

Thus $g_n(x)$ is a polynomial of degree n in x and is the unique solution of (26) that satisfies (37).

Explicit formulas for the generating functions $\Phi(t)$ and $\Psi(t)$ can now be stated without any difficulty.

Here again it would be of interest to explicitly determine the weight functions connected with $\{f_n(x)\}$ and $\{g_n(x)\}$, respectively.

REFERENCE

1. J. Favard, Sur les polynomies de Tchebicheff, Comptes rendus de l' Academie des Sciences, Paris, vol. 200 (1935), pp. 2052-2053.

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