A LOGARITHMIC FORMULA FOR FIBONACCI NUMBERS

Gerard R. Deily
United States Department of Defense
Washington, D.C.

If the logarithm of the Fibonacci number $F_n$ is plotted against $n$, it can be seen that for large $n$ the graph is a straight line. Thus one might expect that Fibonacci numbers could be computed from a formula of the form

$$\log F_n = m \cdot n + b,$$

where $m$ is the slope of the line and $b$ its intersection with the vertical axis. That this is so can easily be shown by manipulating Binet's formula

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

into the following form:

$$F_n = \frac{1}{5} \left( \frac{1 + \sqrt{5}}{2} \right)^n \left[ 1 - \left( \frac{1 - \sqrt{5}}{1 + \sqrt{5}} \right)^n \right]$$

For large $n$, the second term within the bracket becomes negligible, and hence (3) becomes

$$F_n = \frac{1}{5} \left( \frac{1 + \sqrt{5}}{2} \right)^n$$

Taking logarithms then gives

$$\log F_n = n \left[ \log (1 + \sqrt{5}) - \log 2 \right] - \frac{1}{2} \log 5$$

which is of the form (1). If base 10 is used, the characteristic of the logarithm computed in (5) then gives the order of magnitude of $F_n$. This knowledge is useful in determining required sizes of registers when setting up Fibonacci problems for computation.