SOME CONVERGENT RECURSIVE SEQUENCES, HOMEOMORPHIC IDENTITIES, AND INDUCTIVELY DEFINED COMPLEMENTARY SEQUENCES

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I. Consider the following recursive formula for generating sequences, where λ is a given positive number:

(1.1) $$x_{n+1} = x_{n-1} - \lambda x_n \ .$$

It is well known that any sequence so generated is of the form

(1.2) $$x_n = C_1 R_1^n + C_2 R_2^n \ ,$$

where C_1 and C_2 are constants, and R_1 and R_2 are the roots of

(1.3) $$x^2 = 1 - \lambda x \ .$$

Let R_1 be the positive root which is less than one, and let R_2 be the root that is less than minus one. Then this sequence converges if and only if $C_2 = 0$; and when it does converge, it converges to zero. So given two positive numbers x_{-1} and x_0, this sequence converges if and only if $x_0/x_{-1} = R_1$. Furthermore, if λ is an integer, then R_1 is irrational.

It should be noted that sequences generated by a recursive formula such as (1.1) may be continued in the opposite direction. If $\lambda = 1$, then the recursive formula obtained for going in the opposite direction is the formula used for generating Fibonacci numbers.

Let P be a homeomorphism of $[0,\infty)$ onto itself. In other words, $P(0) = 0$ and P is a continuous, unbounded, strictly increasing function of non-negative real numbers. Then the question arises as to the convergence of the sequence starting from two positive numbers x_{-1} and x_0 and generated by the formula

(1.4) $$x_{n+1} = x_{n-1} - P(x_n) \ .$$

Although the properties to be discussed will depend on the nature of P on $[0,\infty)$ only, to facilitate the discussion it will be assumed that P is a homeomorphism of the entire real line. But for this exception, the word homeomorphism as used in this paper will refer to a homeomorphism of $[0, \infty)$ onto itself. Given two homeomorphisms h and g, their sum and products are defined as those homeomorphisms respectively satisfying

$$(1.5) \qquad \begin{aligned} (h + g)(t) &= h(t) + g(t) \\ (hg)(t) &= h[g(t)] \quad \text{and} \\ (gh)(t) &= g[h(t)] \quad \text{for all } t \geq 0. \end{aligned}$$

The inverse h^{-1} of h is that homeomorphism such that

$$(1.6) \qquad hh^{-1} = h^{-1}h = I ,$$

where I is the identity homeomorphism. The relation $h < g$ is defined to mean that $h(t) < g(t)$ for all $t > 0$. Similarly, $h \leq g$ means that $h(t) \leq g(t)$ for all $t > 0$. Note that $h < g$ if and only if $h^{-1} > g^{-1}$ and that $h \leq g$ if and only if $h^{-1} \geq g^{-1}$.

Given two homeomorphisms h and g, define $h \cup g$ as that homeomorphism such that $(h \cup g)(t)$ is the largest of the two numbers $h(t)$ and $g(t)$ for all $t \geq 0$. Define $h \cap g$ as that homeomorphism such that

$$(1.7) \qquad h \cap g + h \cup g = h + g .$$

In other words, $h \cap g$ is the minimum of g and h. Note that for any h and g,

$$(1.8) \qquad h \cap g = g \cap h \leq h \leq h \cup g = g \cup h .$$

Also note that for any homeomorphism h,

$$(1.9) \qquad h \cap h^{-1} \leq I \leq h \cup h^{-1} .$$

The remainder of this first section of this paper is devoted to proving the following five interrelated theorems:

<u>Theorem 1</u>: There exists a unique homeomorphism h such that

$$(1.10) \qquad h = P + h^{-1} .$$

Theorem 2: Let h and g be two homeomorphisms such that

$$h = P + g^{-1} \quad \text{and} \tag{1.11}$$

$$g = P + h^{-1} \ .$$

Then $h = g =$ the homeomorphism of Theorem 1.

Theorem 3: Let g_1 be any homeomorphism. Then the sequence of homeomorphisms $\{g_n\}$ defined inductively by

$$g_{n+1} = P + g_n^{-1} \tag{1.12}$$

converges uniformly on every bounded subset of $[0, \infty)$ to the homeomorphism h of Theorem 1.

Theorem 4: The sequence generated by (1.4) from two positive numbers x_{-1} and x_0 converges if and only if $x_{-1} = h(x_0)$, where h is the homeomorphism of Theorem 1. Also, whenever this sequence converges, it converges to zero.

If $h(x_0) > x_{-1}$, then all of the elements of the sequence with even subscripts are positive, but all but a finite number of the elements with odd subscripts are negative.

If $h(x_0) < x_{-1}$, then all of the odd subscripted elements are positive and all but a finite number of the even subscripted elements are negative.

Theorem 5: If P maps integers into integers, then the h of Theorem 1 will not map any positive integer into an integer.

Proofs: Let h_1 be a homeomorphism such that $h_1 \leq P$. By induction, for n a positive integer, define

$$h_{n+1} = P + h_n^{-1} \ . \tag{1.13}$$

Then $n > 1$ implies that

$$h_1 \leq P < P + h_{n-1}^{-1} = h_n \ . \tag{1.14}$$

By induction on m, if $0 < m < n$ and m is even, then

$$h_m = P + h_{m-1}^{-1} > P + h_{n-1}^{-1} = h_n \tag{1.15}$$

and similarly, if $m \geq 3$ is odd, then

$$h_m = P + h_{m-1}^{-1} < P + h_{n-1}^{-1} = h_n \ . \tag{1.16}$$

So the increasing sequence of homeomorphisms with odd subscripts

$$h_1 < h_3 < h_5 < h_7 < h_9 < --- \tag{1.17}$$

is bounded from above by the decreasing sequence

$$h_2 > h_4 > h_6 > h_8 > h_{10} > --- \ . \tag{1.18}$$

Also, the decreasing sequence

$$h_3^{-1} > h_5^{-1} > h_7^{-1} > h_9^{-1} > --- \tag{1.19}$$

is bounded from below by the increasing sequence

$$h_2^{-1} < h_4^{-1} < h_6^{-1} < h_8^{-1} < --- \ . \tag{1.20}$$

Therefore, these four sequences must be pointwise convergent.

Let us now prove that the homeomorphisms h_n^{-1} for $n > 1$ are uniformly equicontinuous on every bounded subset of $[0, \infty)$. For any $r > 0$, $[0, r]$ is compact and so for any $\epsilon > 0$, there exists a $\delta > 0$ such that $0 \leq P(t_1) < P(t_2) \leq r$ and $P(t_2) - P(t_1) < \delta$ imply that $t_2 < t_1 + \epsilon$.

Let t_1, t_2 and $n > 1$ be such that

$$0 \leq h_n(t_1) < h_n(t_2) \leq r \quad \text{and} \tag{1.21}$$

$$h_n(t_2) - h_n(t_1) < \delta. \tag{1.22}$$

Then

$$0 \leq P(t_1) < P(t_2) < P(t_2) + h_{n-1}^{-1}(t_2) = h_n(t_2) \leq r \ . \tag{1.23}$$

Also

$$(1.24) \qquad P(t_2) - P(t_1) < P(t_2) - P(t_1) + h_{n-1}^{-1}(t_2) - h_{n-1}^{-1}(t_1)$$

$$= h_n(t_2) - h_n(t_1) < \delta .$$

Therefore $t_1 < t_2 < t_1 + \epsilon$. Consequently the h_n^{-1} are uniformly equicontinuous on any set $[0, r]$.

Since the pointwise convergent sequences (1.19) and (1.20) are uniformly equicontinuous on every bounded set $[0, r]$, they converge uniformly to continuous functions. However, sequences (1.17) and (1.18) are related to (1.19) and (1.20) by (1.13). Therefore, these sequences also converge uniformly to continuous functions. But if a series of homeomorphisms and their inverses both converge to continuous functions, then these continuous functions must be homeomorphisms. Therefore, there exist homeomorphisms h and g such that h, g, h^{-1} and g^{-1} are the limits of (1.17), (1.18), (1.19), and (1.20) respectively. Also, (1.13) implies (1.11).

Let h and g be any pair of homeomorphisms which satisfy (1.11). That at least one such pair exists has just been proven. Let x_0 be a positive real number and let $x_{-1} = h(x_0)$. Then let the sequence

$$(1.25) \qquad x_{-1}, x_0, x_1, x_2, x_3, x_4, \cdots$$

be defined inductively by (1.4). Then it will be shown by induction that for $n \geq 0$

$$(1.26) \qquad x_{2n} = (h^{-1}g^{-1})^n (x_0) \quad \text{and}$$

$$(1.27) \qquad x_{2n-1} = h(h^{-1}g^{-1})^n (x_0) \quad .$$

Equations (1.26) and (1.27) are obviously satisfied for $n = 0$ since $(h^{-1}g^{-1})^0$ is defined as I. If they are true for a given n, then

$$(1.28)\qquad \begin{aligned} x_{2(n+1)-1} &= x_{2n-1} - P(x_{2n}) \\ &= \left[h(h^{-1}g^{-1})^n - P(h^{-1}g^{-1})^n\right](x_0) \\ &= (h-P)(h^{-1}g^{-1})^n (x_0) \\ &= g^{-1}(h^{-1}g^{-1})^n (x_0) \\ &= h(h^{-1}g^{-1})^{n+1} (x_0) \ . \end{aligned}$$

and also

$$(1.29)\qquad \begin{aligned} x_{2(n+1)} &= x_{2n} - P\,(x_{2(n+1)-1}) \\ &= (h^{-1}g^{-1})^n (x_0) - Ph(h^{-1}g^{-1})^{n+1} (x_0) \\ &= (g-P)h(h^{-1}g^{-1})^{n+1} (x_0) \\ &= (h^{-1}g^{-1})^{n+1} (x_0) \ . \end{aligned}$$

However, (1.11) implies that

$$(1.30)\qquad h^{-1}g^{-1} < h^{-1}(P + g^{-1}) = h^{-1}h = I \ .$$

Therefore, $(h^{-1}g^{-1})^n (x_0)$ converges to zero as n tends to infinity. Consequently, (1.25) also converges to zero.

Let x_0, y_0, x_{-1} and y_{-1} be any positive numbers such that $x_0 = y_0$ and $y_{-1} - x_{-1} = \varepsilon > 0$. Define the sequence $\{x_n\}$ inductively by (1.4) and likewise the sequence $\{y_n\}$ inductively by

$$(1.31)\qquad y_{n+1} = y_{n-1} - P(y_n) \ .$$

Equations (1.4) and (1.31) yield

$$(1.32)\qquad y_1 - x_1 = y_{-1} - P(y_0) - x_{-1} + P(x_0) = y_{-1} - x_{-1} = \varepsilon \ .$$

Then by induction, for $n > 0$,

(1.33)
$$y_{2n} - x_{2n}$$
$$= y_{2n-2} - x_{2n-2} - P(y_{2n-1}) + P(x_{2n-1})$$
$$< y_{2n-2} - x_{2n-2} \leq 0$$

and

(1.34)
$$y_{2n+1} - x_{2n+1}$$
$$= y_{2n-1} - x_{2n-1} - P(y_{2n}) + P(x_{2n})$$
$$> y_{2n-1} - x_{2n-1} \geq \varepsilon \cdot$$

If $x_{-1} = h(x_0)$, then the y_{2n} terms decrease to less than

(1.35)
$$\lim_{n\to\infty} x_{2n} = 0$$

but the y_{2n+1} terms are bounded above

(1.36)
$$\lim_{n\to\infty} x_{2n+1} + \varepsilon = \varepsilon .$$

Conversely, if $y_{-1} = h(y_0)$, then the x_{2n} terms stay above zero but the x_{2n+1} terms decrease below $-\varepsilon$.

In view of the symmetric roles of h and g in (1.11), it may be similarly shown that the sequence defined by (1.4) converges if and only if $x_{-1} = g(x_0)$. Since this is true for any $x_0 > 0$, it follows that whenever h and g satisfy (1.11) they are the same. Therefore, except for the uniqueness of h, Theorems 1 and 2 have been proven. But the uniqueness of h is also similarly proven since it may likewise be shown that if

(1.37)
$$\hat{h} = P + \hat{h}^{-1}$$

for some homeomorphism $\hat{h}$, then (1.25) converges if and only if $x_{-1} = \hat{h}(x_0)$.

Let g_1 be any homeomorphism for which Theorem 3 is to be tested, and choose

(1.38)
$$h_1 = g_1 \cap g_2 \cap P \ .$$

Then $h_1 \le g_1$ implies by induction that for $n > 0$,

$$g_{2n} = P + g_{2n-1}^{-1} \le P + h_{2n-1}^{-1} = h_{2n} \tag{1.39}$$

and

$$g_{2n+1} = P + g_{2n}^{-1} \ge P + h_{2n}^{-1} = h_{2n+1} \quad . \tag{1.40}$$

Therefore, the sequence $\{g_{2n+1}\}$ is bounded from below by $\{h_{2n+1}\}$ and $\{g_{2n}\}$ is bounded from above by $\{h_{2n}\}$. However, $h_1 \le g_2$ similarly implies that $\{g_{2n+1}\}$ is bounded from above by $\{h_{2n}\}$ and $\{g_{2n}\}$ is bounded from below by $\{h_{2n-1}\}$. $h_1 \le P$ implies that both $\{h_{2n+1}\}$ and $\{h_{2n}\}$ converge uniformly to h on every bounded subset of $[0, \infty)$. Therefore, $\{g_n\}$ also converges uniformly to h on every bounded subset of $[0, \infty)$. Identity (1.12) then implies that $\{g_n^{-1}\}$ also converges uniformly on every such bounded subset.

To prove Theorem 5, note that if P maps integers into integers and that if x_0 and x_{-1} are positive integers such that $h(x_0) = x_{-1}$, then the sequence defined inductively by (1.4) must consist of integers. But

$$x_n = (h^{-1})^n (x_0) \quad \text{for } n \ge 0 \quad . \tag{1.41}$$

implies a slow convergence of $\{x_n\}$ which contradicts the assertion that the elements of the sequence are integers.

II. Sequences defined by

$$x_{n+1} = P(x_n) - x_{n-1} \tag{2.1}$$

are considered in this section of the paper. The homeomorphic identity

$$h + h^{-1} = P \tag{2.2}$$

associated with (2.1) is also discussed here. In order to establish theorems concerning the unique convergence of sequences generated by (2.1) and concerning the existence of solutions to (2.2), additional properties of P will need to be assumed.

Lemma 1: Let h and g be any two homeomorphisms. Then

$$(h \cup g)^{-1} = h^{-1} \cap g^{-1} \tag{2.3}$$

and

$$(h \cap g)^{-1} = h^{-1} \cup g^{-1} \quad . \tag{2.4}$$

Proof: To prove (2.3), it is sufficient to show that $(h \cup g)(x) = y$ implies $(h^{-1} \cap g^{-1})(y) = x$. Whenever

$$g(x) \leqq h(x) = y \quad , \tag{2.5}$$

then

$$h^{-1}(y) = x = g^{-1}g(x) \leqq g^{-1}h(x) = g^{-1}(y) \tag{2.6}$$

and so

$$(h^{-1} \cap g^{-1})(y) = h^{-1}(y) = x \quad . \tag{2.7}$$

Similarly, whenever

$$h(x) \leqq g(x) = y \quad , \tag{2.8}$$

then (2.6) and (2.7) follow when h is replaced by g and g is replaced by h. Hence, (2.3) has been proved.

Replacing h by h^{-1} and g by g^{-1} and applying (2.3) proves (2.4).

Lemma 2: Let h and g be two homeomorphisms such that

$$h + h^{-1} = g + g^{-1} \quad . \tag{2.9}$$

Then

$$h \cup g + (h \cup g)^{-1} = h + h^{-1} \tag{2.10}$$

and

$$h \cap g + (h \cap g)^{-1} = h + h^{-1} \tag{2.11}$$

Proof: The hypothesis (2.9) implies that for every x, $(h \cup g)(x) = h(x)$ if and only if $(h^{-1} \cap g^{-1})(x) = h^{-1}(x)$. Therefore,

$$
\begin{aligned}
(2.12) \qquad & h + h^{-1} \\
&= (h + h^{-1}) \cap (g + g^{-1}) \\
&\leq h \cup g + h^{-1} \cap g^{-1} \\
&\leq (h + h^{-1}) \cup (g + g^{-1}) \\
&= h + h^{-1} \quad .
\end{aligned}
$$

Since the middle term of (2.12) equals $h + h^{-1}$, application of Lemma 1 to it yields (2.10). If h is replaced by h^{-1} and g by g^{-1}, then (2.9) remains invariant. Therefore, if these substitutions are applied to (2.10), the result is also valid. But in view of Lemma 1, this is equivalent to (2.11).

Lemma 3: Let h and g be any two homeomorphisms such that $h \geq g \geq I$. Then for any $x > 0$,

$$
(2.13) \qquad \int_0^x \left[h(t) + h^{-1}(t)\right] dt \geq \int_0^x \left[g(t) + g^{-1}(t)\right] dt
$$

and (2.13) becomes an equality if and only if

$$
(2.14) \qquad g(t) = h(t) \quad \text{for all} \quad h^{-1}(x) \leq t \leq x \quad .
$$

Proof: The set of all points (s, t) such that

$$
\begin{aligned}
(2.15) \qquad & 0 \leq s \leq x \ , \\
& h^{-1}(s) \leq t \leq g^{-1}(s)
\end{aligned}
$$

is the same as the set such that

$$
\begin{aligned}
(2.16) \qquad & 0 \leq s \leq x, \\
& g(t) \leq s \leq h(t) \quad \text{and} \\
& 0 \leq t \leq g^{-1}(x)
\end{aligned}
$$

Therefore (see Figures 1 and 2)

$$(2.17)\qquad \int_0^x [h(t) + h^{-1}(t)]\,dt - \int_0^x [g(t) + g^{-1}(t)]\,dt$$

$$= \int_0^x [h(t) - g(t)]\,dt - \int_0^x [g^{-1}(s) - h^{-1}(s)]\,ds$$

$$= \int_0^x [h(t) - g(t)]\,dt - \int_0^{g^{-1}(x)} [\min(x, h(t)) - g(t)]\,dt$$

$$= \int_{h^{-1}(x)}^{g^{-1}(x)} [h(t) - x]\,dt + \int_{g^{-1}(x)}^{x} [h(t) - g(t)]\,dt \geq 0$$

with equality if and only if $h^{-1}(x) = g^{-1}(x)$ and $h(t) = g(t)$ for $g^{-1}(x) \leq t \leq x$. But these last two conditions together are equivalent to (2.14).

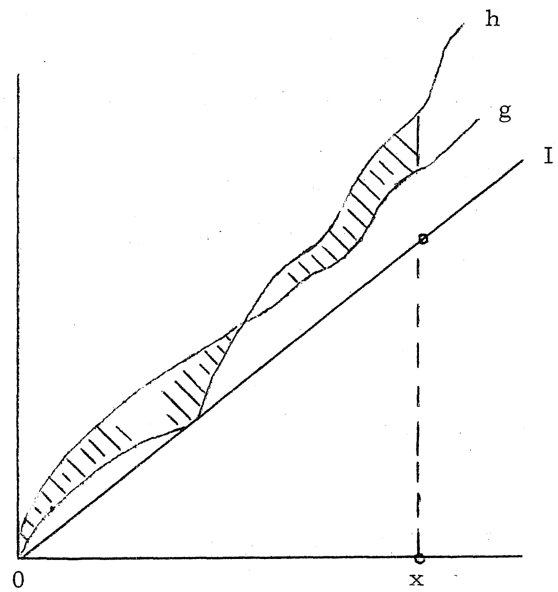

Figure 1: $\int_0^x [h(t) - g(t)]\,dt$ for two typical homeomorphisms h and g such that $h \geq g \geq I$ and $h(x) > g(x)$.

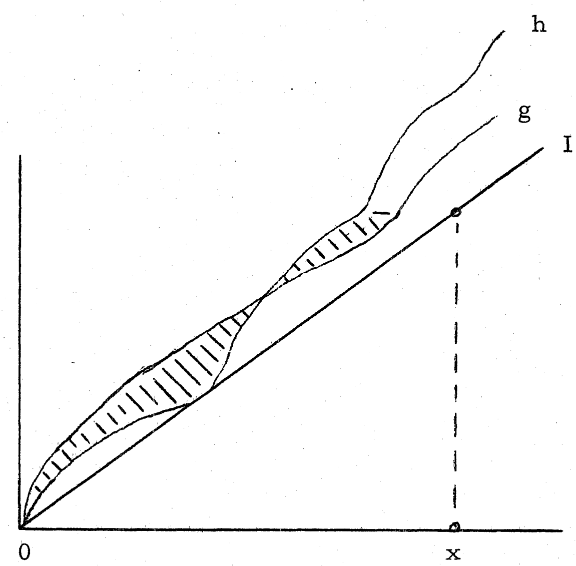

Figure 2: $\int_0^x [g^{-1}(t) - h^{-1}(t)]\, dt$ for two typical homeomorphisms h and g such that $h \geq g \geq I$ and $h(x) > g(x)$.

Theorem 6: Let h and g be any homeomorphisms. Then

(2.9) $$h + h^{-1} = g + g^{-1}$$

if, and only if,

(2.18) $$g(x) = \text{either } h(x) \text{ or } h^{-1}(x) \text{ for all } x \geq 0 \quad .$$

Proof: Define

(2.19) $$f_1 = (h \cup h^{-1}) \cup (g \cup g^{-1}) \quad \text{and}$$

$$f_2 = (h \cup h^{-1}) \cap (g \cup g^{-1}) \quad .$$

Then (1.9) implies that

(2.20) $$f_1 \geq f_2 \geq I \quad .$$

Whenever (2.9) holds, Lemma 2 may be applied four times to yield

$$f_1 + f_1^{-1} = f_2 + f_2^{-1} = h + h^{-1} \quad . \tag{2.21}$$

But integrating (2.21) and applying Lemma 3 to f_1 and f_2 proves that $f_1 = f_2$. Therefore

$$h \cup h^{-1} = g \cup g^{-1} \quad . \tag{2.22}$$

Now Lemma 1 may be applied to obtain

$$h \cap h^{-1} = (h \cup h^{-1})^{-1} = (g \cup g^{-1})^{-1} = g \cap g^{-1} \quad . \tag{2.23}$$

But (2.22) and (2.23) together imply (2.18).

To prove the converse, note that (2.18) implies that $h(x) = x$ if and only if $g(x) = x$. Therefore, the set $\{x \mid h(x) \neq x\}$ is the same set as $\{x \mid g(x) \neq x\}$. Since g and h are both homeomorphisms, each component of this set is mapped homeomorphically onto itself by h and also by g. Furthermore, neither $h-I$ nor $g-I$ changes sign on any such component. So (2.18) implies that, on each component, either $g = h$ and $g^{-1} = h^{-1}$ or else $g = h^{-1}$ and $g^{-1} = h$. Therefore, (2.9) holds on each such component. But (2.9) also holds wherever $h(x) = g(x) = x$.

Corollary: Given any homeomorphism h, there exists one and only one homeomorphism g such that $g \geq I$ and (2.9) $h + h^{-1} = g + g^{-1}$.

Theorem 7: Let h be any homeomorphism. Then for each $x \geq 0$,

$$\int_0^x \left[h(t) + h^{-1}(t)\right] dt \geq x^2 \tag{2.24}$$

and (2.24) becomes an equality if and only if

$$h(x) = x \quad . \tag{2.25}$$

Proof: Lemma 2 implies that

$$h \cup h^{-1} + (h \cup h^{-1})^{-1} = h + h^{-1} \quad . \tag{2.26}$$

Inequality (1.9) implies that the two homeomorphisms $h \cup h^{-1}$ and I satisfy the conditions of Lemma 3. Therefore, (2.24) is established, and (2.24) becomes an equality if and only if

$$(2.27) \qquad (h \cup h^{-1})(t) = t \quad \text{for all} \quad (h \cup h)^{-1}(x) \le t \le x \ .$$

But (2.27) is equivalent to (2.25).

Definition: Given two functions f and g, let either $f \uparrow g$ or $g \downarrow f$ be defined to mean that

$$(2.28) \qquad f(t_2) - f(t_1) \ge g(t_2) - g(t_1)$$

for all t_1 and t_2 on the domains of f and g such that $t_2 > t_1$. Note that if f and g both have continuous derivatives, then this is equivalent to

$$(2.29) \qquad \frac{d}{dt} f(t) \ge \frac{d}{dt} g(t) \quad \text{for all } t \ .$$

Remark: Let α be a positive real number and let f be a continuous function of $[0, \infty)$. Then $f(0) = 0$ and $f \uparrow \alpha I$ if and only if f is a homeomorphism and $f^{-1} \downarrow \alpha^{-1} I$.

Theorem 8: Let $\alpha \ge 2$ and $P \uparrow \alpha I$. Define the sequence of homeomorphisms $\{h_n\}$ inductively by $h_1 = I$ and

$$(2.30) \qquad h_{n+1} = P - h_n^{-1} \quad n > 1 \ .$$

Then the sequence $\{h_n\}$ converges to a homeomorphism h such that

$$(2.2) \qquad h + h^{-1} = P$$

and

$$(2.31) \qquad h \uparrow \left\{ \frac{\alpha + \sqrt{\alpha^2 + 4}}{2} \right\} I \ .$$

Furthermore, the convergence is uniform on every bounded subset of $[0, \infty)$.

Proof: Define by induction, $r_1 = 1$ and $r_{n+1} = \alpha - r_n^{-1}$. Then $h_1 \uparrow r_1 I$ and by induction, (2.32) $$h_n^{-1} \downarrow r_n^{-1} I$$

and

(2.33) $$h_{n+1} = P - h_n^{-1} \uparrow \alpha I - r_n^{-1} I = r_{n+1} I$$

and the h_n are all homeomorphisms.

Also,

(2.34) $$h_2 = P - I \geq I = h_1$$

and so by induction,

(2.35) $$h_{n+2} = P - h_{n+1}^{-1} \geq P - h_n^{-1} = h_{n+1} \quad .$$

Since for each $x \geq 0$, $h_n(x)$ is a monotonic non-decreasing sequence of numbers bounded above by $P(x)$, the sequence $\{h_n\}$ is pointwise convergent. Since the r_n are increasing, (2.32) implies that the h^{-1} are uniformly equicontinuous on every bounded subset of $[0,\infty)$. But this combined with (2.30) implies that the h_n are also uniformly equicontinuous on each such subset. Therefore, the sequence $\{h_n\}$ converges uniformly on every bounded subset of $[0,\infty)$ to some homeomorphism h.

Since $\{r_n\}$ is increasing but bounded by α, it must converge to some number r such that $1 < r \leq \alpha$. By continuity, $r = \alpha - r^{-1}$. Therefore, $h \uparrow r I$ which is the same as (2.31).

<u>Theorem 9</u>: In addition to the hypothesis to Theorem 8, let $P \downarrow \beta I$ where β is some real number $\geq \alpha$.

Then

(2.36) $$h \downarrow \left(\frac{\beta + \sqrt{\beta^2 + 4}}{2} \right) I \ ,$$

where h is the homeomorphism to which the sequence of homeomorphisms of Theorem 8 converge.

<u>Proof</u>: Define $v_1 = 1$ and by induction

(2.37) $$v_{n+1} = \beta - v_n^{-1} \quad .$$

Then by induction

$$h_{n+1} \downarrow P - h_n^{-1} \downarrow \beta I - v_n^{-1} I = v_{n+1} I \quad . \tag{2.38}$$

Therefore, $h \downarrow v I$, where

$$v = \lim_{n \to \infty} v_n = \left(\frac{\beta + \sqrt{\beta^2 + 4}}{2} \right) . \tag{2.39}$$

<u>Corollary:</u> Let $P = \alpha I$. Then

$$h = \left(\frac{\alpha + \sqrt{\alpha^2 + 4}}{2} \right) I \quad . \tag{2.40}$$

<u>Lemma 4:</u> Let

$$h + h^{-1} = P \tag{2.2}$$

and let x_0 and x_{-1} be two positive real numbers such that

$$x_0 < h(x_0) = x_{-1} \tag{2.41}$$

Then the sequence $\{x_n\}$ defined inductively by

$$x_{n+1} = P(x_n) - x_{n-1} \tag{2.1}$$

will converge monotonically to y, where y is the largest real number such that

$$h(y) = y < x_0 \quad . \tag{2.42}$$

However, for no $n > 0$ is $x_n = y$.

<u>Proof:</u> For $n = -1$ or 0, we have that $x_n = h^{-n}(x_0)$, where h^0 is defined as I. Therefore, by induction, for $n > 0$,

$$\begin{aligned} x_{n+1} &= P(x_n) - x_{n-1} \\ &= h(x_n) + h^{-1}(x_n) - x_{n-1} \\ &= h^{-n+1}(x_0) + h^{-n-1}(x_0) - h^{-n+1}(x_0) \\ &= h^{-n-1}(x_0) \quad . \end{aligned} \tag{2.43}$$

Since $h^{-1}(x_0) < x_0$, the sequence $\{x_n\}$ must converge to y as described.

Theorem 10: Let h be any homeomorphism such that $h + h^{-1}$ maps positive integers into integers. Then h will never map any positive integer p into an integer unless $h(p) = p$.

Proof: If the theorem is false, then there exist positive integers p and q such that

$$hUh^{-1}(p) = q > p \quad . \tag{2.44}$$

Lemma 2 implies that

$$hUh^{-1} + (hUh^{-1})^{-1} = h + h^{-1} \quad . \tag{2.45}$$

Define x_0 as p and x_{-1} as q and define the sequence $\{x_n\}$ inductively by

$$x_{n+1} = h(x_n) + h^{-1}(x_n) - x_{n-1} \quad . \tag{2.46}$$

Then applying Lemma 4 to hUh^{-1}, one may see that $\{x_n\}$ must slowly converge as described in the lemma. However, this contradicts the fact, which may be easily verified by induction, that the sequence $\{x_n\}$ consists of integers.

Theorem 11: Let $P \uparrow 2I$ and $P > 2I$ on $(0, \infty)$. Let x_0 and x_{-1} be two positive real numbers.

Then a necessary and sufficient condition that the sequence defined inductively by

$$x_{n+1} = P(x_n) - x_{n-1} \tag{2.1}$$

converges is that $h(x_0) = x_{-1}$, where h is the unique homeomorphism such that $h \geq I$ and

$$h + h^{-1} = P \quad . \tag{2.2}$$

The sequence will contain a non-positive element if and only if $h(x_0) < x_{-1}$. Also, $x_{n+1} > x_n$ for some element if and only if $h(x_0) > x_{-1}$, and this holds if and only if

(2.47) $$\lim_{n \to \infty} x_n = +\infty .$$

Proof: The existence and uniqueness of h is given by Theorem 8 and the corollary to Theorem 6. $P > 2I$ and (2.2) imply that $h > I$. If $x_{-1} = h(x_0)$, then Lemma 4 implies that x_n converges monotonically to zero.

Let x_0, y_0, x_{-1} and y_{-1} be positive numbers such that $x_0 = y_0$ and $y_{-1} - x_{-1} = \varepsilon > 0$. Define $\{x_n\}$ inductively by (2.1) and likewise $\{y_n\}$ by

(2.48) $$y_{n+1} = P(y_n) - y_{n-1} .$$

If $n = -1$, then

(2.49) $$x_n - y_n \geq n\,\varepsilon$$

and

(2.50) $$(x_{n+1} - y_{n+1}) - (x_n - y_n) \geq \varepsilon .$$

Therefore, by induction, for $n \geq 0$,

(2.50) $$\begin{aligned} &(x_{n+1} - y_{n+1}) - (x_n - y_n) \\ &= P(x_n) - P(y_n) - (x_n - y_n) - (x_{n-1} - y_{n-1}) \\ &\geq (x_n - y_n) - (x_{n-1} - y_{n-1}) \geq \varepsilon \end{aligned}$$

and

$$\begin{aligned} &x_n - y_n \\ &= (x_n - y_n) - (x_{n-1} - y_{n-1}) + (x_{n-1} - y_{n-1}) \\ &\geq \varepsilon + (x_{n-1} - y_{n-1}) \\ &\geq \varepsilon + (n-1)\,\varepsilon = n\,\varepsilon . \end{aligned}$$

If $y_{-1} = h(y_0)$, then (2.49) implies that x_n is always positive for $n \geq 0$ and it converges to infinity. If $x_{-1} = h(x_0)$, then (2.49) implies that y_n is monotonic and will attain negative values.

III. Let a and b be monotonic increasing mappings of positive integers into positive integers. Then the sequences $\{a(n)\}$ and $\{b(n)\}$ are said to be complementary if and only if each positive integer is represented in one and only one of these sequences.

Given a real number r, define $[r]$ as the integer part of r, namely $[r]$ is that integer such that

$$[r] \le r < [r] + 1 \quad (3.1)$$

Define $[r]^*$ as that integer such that

$$[r]^* < r \le [r]^* + 1 , \quad (3.2)$$

or equivalently

$$[r]^* = -1 - [-r] . \quad (3.3)$$

A result of S. Beatty, see Reference [1], is essentially that given a positive irrational number x, then the sequences $[(1 + x)\, n]$ and $[(1 + x^{-1})\, n]$ are complementary. This result has since turned up many times in the literature, often in the form that if α and β are two positive irrational numbers such that $\alpha^{-1} + \beta^{-1} = 1$, then the two sequences $[\alpha n]$ and $[\beta n]$ are complementary.

A generalization of this result by Lambek and Moser states that the sequences $\{a(n)\}$ and $\{b(n)\}$ are complementary if and only if for each pair of positive integers m and n, either $a(m) - m < n$ or else $b(n) - n < m$ but never both. This result combined with Lemma 5 may also be used to prove Theorem 12 instead of the proof given.

Lemma 5: Let f and g be homeomorphisms such that

$$f^{-1} + g^{-1} = I . \quad (3.4)$$

Then $f - I$ and $g - I$ are homeomorphisms and

$$(f - I)(g - I) = I . \quad (3.5)$$

Conversely, let h be any homeomorphism. Then

$$(I + h)^{-1} + (I + h^{-1})^{-1} = I . \quad (3.6)$$

Proof:

(3.7) $$f - I = (I - f^{-1})\, f = g^{-1} f \quad \text{and}$$

$$g - I = (I - g^{-1})\, g = f^{-1} g \quad .$$

But $g^{-1} f$ and $f^{-1} g$ are homeomorphisms which are inverses of each other.

(3.8) $$(I + h)^{-1} + (I + h^{-1})^{-1}$$

$$= (I + h)^{-1} + (I + h^{-1})^{-1} (h + I)\, h^{-1} h\, (I + h)^{-1}$$

$$= (I + h)^{-1} + (I + h^{-1})^{-1} (I + h^{-1})\, h\, (I + h)^{-1}$$

$$= (I + h)^{-1} + h\, (I + h)^{-1}$$

$$= (I + h)\, (I + h)^{-1} = I \quad .$$

Theorem 12: Let f and g be two homeomorphisms such that

(3.4) $$f^{-1} + g^{-1} = I \quad .$$

Then the two sequences $\{[f(n)]\}$ and $\{[g(n)]^*\}$ are complementary.

Proof: Given a non-negative integer m, let n_1 be the number of elements of $\{[f(n)]\}$ which are less than or equal to m, and let n_2 be the number of such elements of $\{[g(n)]^*.\}$ Then

(3.9) $$f(n_1) < m + 1 \le f(n_1 + 1) \quad \text{and}$$

$$g(n_2) \le m + 1 < g(n_2 + 1) \quad .$$

Applying f^{-1} and g^{-1} to (3.9) yields

(3.10) $$n_1 = f^{-1} f(n_1) < f^{-1}(m + 1) \le f^{-1} f(n_1 + 1) = n_1 + 1$$

$$n_2 = g^{-1} g(n_2) \le g^{-1}(m + 1) < g^{-1} g(n_2 + 1) = n_2 + 1 \quad .$$

Adding the two parts of (3.10) together yields

(3.11) $$n_1 + n_2 < m + 1 < n_1 + n_2 + 2 \quad .$$

Since n_1, n_2 and m are all integers, it follows that

(3.12) $$n_1 + n_2 = m \;.$$

Therefore, each positive integer is represented once and only once by the sequences, but only positive integers are represented.

Corollary: Let h be any homeomorphism. Then the sequences $n + [h^{-1}(n)]^*$ and $n + [h(n)]$ are complementary.

Proof: Apply Lemma 5 to the theorem.

The analysis of Wythoff's game (see Reference [4]) involves complementary sequences $\{a(n)\}$ and $\{b(n)\}$ such that

(3.13) $$b(n) = a(n) + n \;.$$

In a later paper, a generalization of Wythoff's game will be given for which the analysis will involve complementary sequences such that

(3.14) $$b(n) = a(n) + (k+1)\,n,$$

where k is some non-negative integer which defines the game. Beatty's result is easily used to show that the complementary sequences satisfying (3.14) are

(3.15) $$a(n) = \left[\left(\frac{1-k+\sqrt{(k+1)^2+4}}{2}\right) n\right] \quad \text{and}$$

$$b(n) = \left[\left(\frac{3+k+\sqrt{(k+1)^2+4}}{2}\right) n\right]$$

Theorem 13 may be thought of as a generalization of this result.

Theorem 13: Let P map integers into integers. Let the sequences be defined inductively as follows:

(3.16) $$a(1) = 1$$

(3.17) $$b(n) = a(n) + P(n) \qquad n > 0$$

$a(n+1)$ = smallest integer not represented by either $a(i)$

(3.18) or $b(i)$ for some $i \le n$.

Then

$$a(n) = n + [h^{-1}(n)] \quad \text{and} \tag{3.19}$$

$$b(n) = n + [h(n)] ,$$

where h is the unique homeomorphism such that

$$h = P + h^{-1} . \tag{1.10}$$

<u>Proof:</u> Theorem 5 implies that

$$[h(n)]^* = [h(n)] . \tag{3.20}$$

So the corollary to Theorem 12 shows that the sequences defined by (3.19) are complementary. Since (1.10) implies that $h > h^{-1}$, (3.18) and (3.16), which is a special case of (3.18), are satisfied. Equation (3.17) follows from (1.10) and (3.19) and the fact that the P(n) are integers.

In Reference [4] is presented the following result: Let k be an integer greater than 4. Then the sequences defined by

$$a(n) = \left[\left(\frac{k-\sqrt{k^2-4k}}{2}\right)n\right] \quad \text{and} \tag{3.21}$$

$$b(n) = \left[\left(\frac{k+\sqrt{k^2-4k}}{2}\right)n\right]$$

are the sequences such that for n any positive integer,

$$a(n) + b(n) = nk-1 \tag{3.22}$$

and such that a(n) is the smallest positive integer not represented by any a(i) or b(i) with $i < n$.

The following theorem and its corollary may be thought of as generalizations of this result since they imply it with the help of the corollary to Theorem 9.

<u>Theorem 14:</u> Let P map integers into integers. Let there exist a homeomorphism h satisfying

$$h + h^{-1} = P . \tag{2.2}$$

Let the sequences $\{a(n)\}$ and $\{b(n)\}$ be defined inductively as follows: $a(n)$ is the smallest positive integer not represented by earlier elements of a and b, and

$$b(n) = P(n) + 2n - 1 - a(n) \quad . \tag{3.23}$$

Then no positive integer will be represented twice by the two sequences and for each $n > 0$,

$$a(n) = n + [h^{-1}(n)]^* \quad \text{and}$$
$$b(n) = n + [h(n)] \quad , \tag{3.24}$$

where h is the unique homeomorphism such that $h \geq I$ and (2.2) is valid.

Proof: The sequences defined by (3.24) are complementary by the corollary to Theorem 12. Since h satisfies (2.2), the sequences defined by (3.24) satisfy (3.23). Finally, monotonicity of the sequences, their being complementary and the fact that $h^{-1} \leq h$ imply that $a(n)$ is the first such integer not previously represented.

Corollary: Let $P(n) \neq 2n$ for any integer $n > 0$. Then

$$a(n) = n + [h^{-1}(n)] \quad . \tag{3.25}$$

Proof: Theorem 10 implies that $[h^{-1}(n)]^* = [h^{-1}(n)]$.

Theorem 15: Let $\{a(n)\}$ and $\{b(n)\}$ be the sequences of Theorem 13. Let x_0 and x_{-1} be any two positive integers. Let the sequence $\{x_n\}$ be inductively defined by

$$x_{n+1} = x_{n-1} - P(x_n) \tag{1.4}$$

Then the first element of this sequence of integers to be non-positive will have an even subscript if and only if

$$x_0 \leq a(x_{-1}) - x_{-1} \tag{3.26}$$

which in turn is equivalent to

$$x_{-1} > b(x_0) - x_0 \quad . \tag{3.27}$$

Proof: Theorem 5 implies that $h(x_0) \neq x_{-1}$. Hence (3.19) implies that (3.26) and (3.27) are both equivalent to $x_{-1} > h(x_0)$. The proof is now completed by applying Theorem 4.

This theorem may also be proven by the results of Lambek and Moser (Reference [6]).

Theorem 16: Let P map integers into integers and $P \uparrow 2I$ and $P > 2I$ on $[0, \infty)$ and let a(n) and b(n) be the sequences of Theorem 14. Let x_0 and x_{-1} be any two positive integers. Let the sequence x_n be inductively defined by

$$x_{n+1} = P(x_n) - x_{n-1} \quad (2.1)$$

Then the following four statements are logically equivalent:

$$x_0 \leq a(x_{-1}) - x_1 \quad (3.26)$$

$$x_{-1} > b(x_0) - x_0 \quad (3.27)$$

$$\{x_n\} \text{ contains a non-positive element} \quad (3.28)$$

$$\{x_n\} \text{ is monotonic decreasing}$$

Proof: Theorem 10 implies that $h(x_0) \neq x_{-1}$. Hence (3.24) implies that (3.26) and (3.27) are both equivalent to $x_{-1} > h(x_0)$. The proof is now completed by application of Theorem 11.

IV. In this section, representations are sought for homeomorphisms and corresponding complementary sequences associated with P's such that

$$P(n) = 2\alpha n + 2\beta \quad (4.1)$$

for n a positive integer. The numbers 2α and 2β are assumed to be integer constants. The requirement that P be a homeomorphism leads to the conditions that $\alpha > 0$ and

$$\alpha + \beta = \frac{1}{2} P(1) > 0. \quad (4.2)$$

Example 1: For this example, let the function F be defined as

$$F(x) = (\sqrt{\alpha^2+1} - \alpha)\, x - \beta + (\sqrt{\alpha^2+1} - 1)\, \beta/\alpha \ . \quad (4.3)$$

The inverse of this function is

(4.4) $$F^{-1}(x) = (\sqrt{\alpha^2+1} + \alpha)\, x + \beta + (\sqrt{\alpha^2+1} - 1)\, \beta/\alpha \ .$$

Define

(4.5) $$h^{-1}(x) = xF(1) \qquad 0 \le x \le 1$$

$$h^{-1}(x) = F(x) \qquad x \ge 1$$

For h^{-1} to be a homeomorphism, it is necessary that $F(1) > 0$. With some algebraic manipulation, it is readily seen that this requirement is equivalent to

(4.6) $$(\alpha + 2\beta)(1-\beta) > \alpha\beta \qquad \text{or}$$

$$(\alpha + \beta)(1-2\beta) > -\beta \ .$$

By utilizing (4.2), it is seen that (4.6) is satisfied if and only if $\beta \le 1/2$. Condition (4.2) and the requirement that $\alpha > 0$ imply that $h^{-1}(1) < 1$. Therefore,

(4.7) $$h(x) = F^{-1}(x) \qquad x \ge 1$$

and so for n a positive integer,

(4.8) $$h(n) = 2\alpha n + 2\beta + h^{-1}(n) \ .$$

So Theorems 5 and 13 give that the sequences

(4.9) $$a(n) = n + [F(n)] \qquad \text{and}$$

$$b(n) = n + [F^{-1}(n)]$$

are complementary and satisfy

(4.10) $$b(n) = 2\alpha n + 2\beta + a(n) > a(n)$$

unless $\beta \ge 1$. In the case where $\beta \ge 1$, other representations are needed. Setting $\beta = 0$ and $\alpha = (k+1)/2$ yields (3.15).

For the next two examples, a homeomorphism $h \ge I$ is sought such that for n a positive integer

(4.11) $$h(n) + h^{-1}(n) = 2\alpha n + 2\beta \ .$$

However, in some cases of Example 3, a homeomorphism is found that only generates the complementary sequences that would be generated by a homeomorphism satisfying (4.11). In these cases, a representation of these sequences is obtained.

Theorem 7 implies that $\alpha \geq 1$. Since the sum of two unequal positive integers is at least three, Theorem 14 implies that

$$\alpha + \beta = ([h(1)] + [h^{-1}(1)]^* + 1)/2 \geq 1 \tag{4.12}$$

Example 2: For this example, let $\beta \leq 0$ and let the function F be defined as

$$F(x) = (\alpha \sqrt{\alpha^2-1})\, x + \beta - \beta \sqrt{\alpha^2-1}/(\alpha-1) \ . \tag{4.13}$$

If $\alpha = 1$, then $\beta = 0$ and let the last term of (4.13), which would be indeterminate, be assumed to vanish. The inverse of this function is

$$F^{-1}(x) = (\alpha + \sqrt{\alpha^2-1})x + \beta + \beta \sqrt{\alpha^2-1}/(\alpha-1) \ . \tag{4.14}$$

Let h^{-1} be defined according to (4.5) with this F being used instead of the F of Example 1. The conditions on α and β imply that

$$0 < F(x) \leq x \qquad x \geq 1 \tag{4.15}$$

Therefore, for $x \geq 1$, $h(x) = F^{-1}(x)$ and so (4.11) is satisfied for any positive integer n.

If $\alpha > 1$ and $\beta = 0$, then application of the corollary to Theorem 14 yields Ky Fan's result summarized by (3.21) and (3.22). If $\alpha = 1$ and $\beta = 0$, then $h = I$ and the resulting complementary sequences are represented by

$$a(n) = 2n - 1 \quad \text{and} \qquad b(n) = 2n \ . \tag{4.16}$$

Example 3: For this example, let $\beta \geq 1/2$ and let the function F be defined for $x \geq 1$ as

$$F(x) = \alpha x + \beta - \sqrt{(\alpha x+\beta)^2 - (x-\beta)^2 - \varepsilon} = \alpha x + \beta - \sqrt{(\alpha+1)(\alpha x^2 - x^2 + 2\beta x) - \varepsilon} \tag{4.17}$$

where ε is a constant to be appropriately chosen. The inverse to this function is often two-valued. Considering only the largest of these two values yields

(4.18) $$F^{-1}(x) = \alpha x + \beta + \sqrt{(\alpha x+\beta)^2 - (x-\beta)^2 - \varepsilon}$$

It may be shown that $dF(x)/dx > 0$ if and only if

(4.19) $$(x+\beta)\left\{(\alpha-1)x + \beta\alpha + \beta\right\} > \alpha^2\varepsilon/(\alpha+1) .$$

Furthermore, if $\varepsilon = 0$, then $F(x)$ is positive for all $x > \beta$. So for the case where $\beta = \frac{1}{2}$, set $\varepsilon = 0$ and define h^{-1} with this F according to (4.5).

For this case, $0 < F(1) < 1$. Therefore

(4.20) $$h(x) = F^{-1}(x) \qquad x \geq 1 .$$

Application of Theorem 14 and its corollary imply that the sequences defined by

(4.21) $$a(n) = \left[(\alpha+1)n + \frac{1}{2} - \sqrt{(\alpha^2-1)n^2 + (\alpha+1)n}\right] \quad \text{and}$$

$$b(n) = \left[(\alpha+1)n + \frac{1}{2} + \sqrt{(\alpha^2-1)n^2 + (\alpha+1)n}\right]$$

are complementary and that

(4.22) $$a(n) < b(n) = 2(\alpha+1)n - a(n) .$$

In the paper that will generalize Wythoff's game, related games will be presented whose analysis utilizes these two complementary sequences.

For $\beta = 1$, choose $\varepsilon > 0$ but sufficiently small that

(4.23) $$0 < F(1) < F(2) < 1$$

and that (4.19) is satisfied for $x \geq 2$.

Define

(4.24) $$h^{-1}(x) = xF(x) \qquad 0 \leq x \leq 1$$

$$h^{-1}(x) = (2-x)F(1) + (x-1)F(2) \qquad 1 \leq x \leq 2$$

$$h^{-1}(x) = F(x) \qquad x > 2 .$$

Then $h(x) = F^{-1}(x)$ for $x \geq 1$. Consequently, h will satisfy (4.11) for this case.

If $\beta \geq 1\frac{1}{2}$, then $F(1) > 0$ implies that $\varepsilon < (\beta-1)^2$. It may be shown that this in turn implies that $F(1) > F(2\beta-1)$. Consequently, there does not exist any homeomorphism which equals F for positive integers. However, for certain cases, homeomorphisms will be defined such that

$$[h^{-1}(n)]^* = [F(n)]^* = [F(n)] \quad \text{and} \tag{4.25}$$

$$h(n) = F^{-1}(n).$$

So in these cases, the sequences defined by (4.9) are complementary and satisfy

$$a(n) < b(n) = 2(\alpha+1)n + 2\beta - 1 - a(n) \ . \tag{4.26}$$

If $2\beta \geq 3$ is odd, then the requirement that $F(\beta \pm \frac{1}{2})$ be positive implies that $\varepsilon > -1/4$. If $2\beta \geq 4$ is even, then positivity of $F(\beta)$ implies that $\varepsilon > 0$. For the sequences defined by (4.9) to be monotonic and complementary, it is necessary that

$$[F(1)] = a(1) - 1 = 0 \ . \tag{4.27}$$

The requirement that $F(1) < 1$ is equivalent to

$$2(\alpha+1) - (\beta-2)^2 > \varepsilon \ . \tag{4.28}$$

If 2β is odd, then the left side of (4.28) equals 3/4 modulo one, but if 2β is even, the left side is an integer. Therefore, a necessary condition for the attainment of the present objectives is that

$$2(\alpha+1) > (\beta-2)^2 \tag{4.29}$$

This condition will also turn out to be sufficient. Furthermore, to attain these objectives when (4.29) is valid, it is sufficient that

$$0 < \varepsilon < \frac{1}{4} \ . \tag{4.30}$$

Condition (4.30) implies that (4.19) is satisfied whenever $x \geq \beta + \frac{1}{2}$. Also, (4.30) may be shown to imply that

$$F(\beta+1) < 1 \ . \tag{4.31}$$

Define

(4.32) $$h^{-1}(x) = xF([\beta+1])/[\beta+1] \qquad 0 \le x \le [\beta+1]$$

$$h^{-1}(x) = F(x) \qquad x \ge [\beta+1] .$$

Then h^{-1} is a homeomorphism, and since

(4.33) $$h^{-1}([\beta+1]) < 1 ,$$

it follows that

(4.34) $$h(x) = F^{-1}(x) \qquad x \ge 1$$

Condition (4.30) implies that $F(n)$ can never be a multiple of $\frac{1}{2}$ for any integer n. For any $1 \le x \le [\beta+1]$, both $h^{-1}(x)$ and $F(x)$ are between zero and one. Therefore, (4.25) is satisfied for all positive integers as desired.

V. The purpose of this section is to generalize the results of the first section. Whereas the proof of Theorem 17 uses ideas not found in the first section, the remainder of this section utilizes mostly straightforward generalizations of the techniques of Section I plus applications of Theorem 17. In this section, μ always refers to a positive real number and μ^{-1} to its reciprocal.

Theorem 17: Let $\mu \le 1$ and let h and g be two homeomorphisms such that

(5.1) $$h + \mu g^{-1} = g + \mu h^{-1} .$$

Let $h(x) \neq g(x)$ for some $x > 0$. Then $h(t) > t$ for all $t \ge x$.

Proof: Given any point $t > 0$, if $h(t) > g(t)$, then

(5.2) $$h^{-1}h(t) = g^{-1}g(t) < g^{-1}h(t)$$

which by (5.1) implies that

(5.3) $$hh(t) > gh(t)$$

Similarly, $h(t) < g(t)$ implies (5.3) with the inequality reversed.

If the theorem is false, then for some point x, $h(x) \neq g(x)$ and either

$$h(x) \leq x \tag{5.4}$$

or else

$$x < h(x) < x_0 = h(x_0) \tag{5.5}$$

for some point x_0. Define x_1 as either x or $h(x)$, whichever satisfies

$$h(x_1) < g(x_1) \ . \tag{5.6}$$

For $n > 1$, define x_n as $h^{n-1}(x_1)$. Then in case of (5.4), the sequence $\{x_n\}$ is monotonic non-increasing and bounded below by zero. In case of (5.5), $\{x_n\}$ is monotonic increasing and bounded above by x_0.

The first paragraph of this proof implies that for $n > 0$,

$$h(x_{2n-1}) < g(x_{2n-1}) \quad \text{and} \tag{5.7}$$

$$h(x_{2n}) > g(x_{2n}) \ .$$

Let (y_n, z_n) be the component of $\{t \mid h(t) \neq g(t)\}$ which contains x_n. Then the first paragraph of this proof implies that the open intervals (y_n, z_n) are all disjoint and that for $n > 0$

$$y_{n+1} = h(y_n) = g(y_n) \quad \text{and} \tag{5.8}$$

$$z_{n+1} = h(z_n) = g(z_n) \ .$$

When (5.4) holds, then

$$z_{n+1} \leq y_n < x_n < z_n \ , \tag{5.9}$$

and when (5.5) holds, then

$$y_n < x_n < z_n \leq y_{n+1} < x_0 \ . \tag{5.10}$$

Integration by parts and (5.8) imply that

(5.11) $$(-\mu)^n \int_{y_{n+1}}^{z_{n+1}} \left\{ g(t) - h(t) \right\} dt$$

$$= -(-\mu)^n \int_{y_{n+1}}^{z_{n+1}} \left\{ t dg(t) - t dh(t) \right\}$$

which replacing g(t) by u and h(t) by v

$$= -(-\mu)^n \left\{ \int_{y_n}^{z_n} g^{-1}(u) du - \int_{y_n}^{z_n} h^{-1}(v) dv \right\}$$

$$= (-\mu)^{n-1} \int_{y_n}^{z_n} \left\{ g(t) - h(t) \right\} dt$$

which by induction on n

$$= \int_{y_1}^{z_1} \left\{ g(t) - h(t) \right\} dt > 0 .$$

In case of (5.4), define x_0 as z_1. Then for either case,

(5.12) $$\int_0^{x_0} \left\{ h \cup g - h \cap g \right\} (t) \, dt$$

$$\geq \sum_{n=1}^{\infty} \int_{y_n}^{z_n} \left\{ h \cup g - h \cap g \right\} (t) \, dt$$

$$= \sum_{n=1}^{\infty} \mu^{1-n} \int_{y_1}^{z_1} \left\{ g(t) - h(t) \right\} dt = \infty$$

which is impossible.

Corollary: Let $\mu \geq 1$ and let h and g satisfy (5.1). Then $h(x) \neq g(x)$ implies that $h(t) < t$ for all $t \geq x$.

Proof: Replace h by h^{-1}, g by g^{-1} and μ by μ^{-1} and apply the theorem.

Theorem 18: Let h and g be two-homeomorphisms such that

$$(5.13) \qquad h + g^{-1} = g + h^{-1} .$$

Then $h = g$.

Proof: Use Theorem 17 and its corollary.

Theorem 19: Let h_1 be a homeomorphism such that $h_1 \leq P$. Define by induction for $n > 0$,

$$(5.14) \qquad h_{n+1} = P + \mu h_n^{-1} .$$

Then on each bounded subset of $[0,\infty)$, $\{h_{2n-1}\}$ and $\{h_{2n}\}$ converge uniformly to homeomorphisms h and g respectively. Furthermore, $h \leq g$ and

$$(5.15) \qquad h = P + \mu g^{-1} \quad \text{and}$$

$$g = P + \mu h^{-1} .$$

Proof: The arguments of (1.13) through (1.24) and the next paragraph remain unchanged except that h_{n-1}^{-1} is replaced by μh_{n-1}^{-1} and h_{m-1}^{-1} is replaced by μh_{m-1}^{-1}.

Theorem 20: Let $\mu \leq 1$ and x_0 be a positive number. Then there cannot exist more than one positive number x_{-1} such that the sequence x_n defined inductively by

$$(5.16) \qquad x_{n+1} = \mu^{-1} x_{n-1} - \mu^{-1} P(x_n)$$

converges.

Proof: Let y_0, x_{-1}, ε and y_{-1} be positive numbers such that $y_0 = x_0$ and $y_{-1} - x_{-1} = \varepsilon$.

Define $\{y_n\}$ inductively by

$$y_{n+1} = \mu^{-1} y_{n-1} - \mu^{-1} P(y_n) \tag{5.17}$$

Then analogous to (1.32), (1.33) and (1.34) are the similarly obtained results

$$y_1 - x_1 = \mu^{-1} \varepsilon \tag{5.18}$$

and

$$y_{2n} - x_{2n} < \mu^{-1}(y_{2n-2} - x_{2n-2}) \leq 0 \tag{5.19}$$

and

$$y_{2n+1} - x_{2n+1} > \mu^{-1}(y_{2n-1} - x_{2n-1}) \geq \mu^{-n-1} \varepsilon \geq \varepsilon . \tag{5.20}$$

So at most one of the sequences may converge.

Theorem 21: Let h and g be two homeomorphisms such that

$$h = P + \mu h^{-1} \tag{5.21}$$

and

$$g = P + \mu g^{-1} . \tag{5.22}$$

Then $h = g$.

Proof: Identities (5.21) and (5.22) imply (5.1). If $\mu \geq 1$, then (5.21) implies that $h > h^{-1}$. Therefore, $h > I$ and the corollary to Theorem 17 finishes the proof for $\mu \geq 1$.

If $\mu < 1$, then for any $x > 0$, either $h(x) \leq x$ or $g(x) \leq x$ or else $h(x) > x$ and $g(x) > x$. In the first two cases Theorem 17 implies that $h(x) = g(x)$. In the last case, the sequences $\{x_n\} = \{h^{-n}(x)\}$ and $\{y_n\} = \{g^{-n}(x)\}$ are both convergent. But these sequences satisfy (5.16) and (5.17). Since $x_0 = y_0 = x$, Theorem 20 implies that

$$h(x) = x_{-1} = y_{-1} = g(x) . \tag{5.23}$$

Theorem 22: Let $\mu \leq 1$ and $P + I > I$. Let g_1 be any homeomorphism. Then the sequence of homeomorphisms defined inductively by

$$g_{n+1} = P + \mu g_n^{-1} \tag{5.24}$$

converges uniformly on every bounded subset of $[0, \infty)$ to a unique homeomorphism h such that

$$h = P + \mu h^{-1} . \tag{5.21}$$

Proof: Choose

$$h_1 = g_1 \cap g_2(\mu I) \cap P . \tag{5.25}$$

Then Theorem 19 is applicable. For any $x > 0$, $g(x) > x$ for if not, $h(x) \leq g(x) \leq x$ which implies that $h^{-1}(x) \geq x$ and hence that

$$g(x) = P(x) + \mu h^{-1}(x) \geq P(x) + \mu x > x . \tag{5.26}$$

Therefore,

$$hg = (P + \mu g^{-1})g = Pg + \mu I > P + \mu I > I . \tag{5.27}$$

But $hg > I$ implies that $h > g^{-1}$ which in turn implies that $gh > I$.

For any point $x_0 > 0$, define x_{-1} as $h(x_0)$. Let the sequence $\{x_n\}$ be defined inductively by (5.16). Then arguments analogous to (1.28) and (1.29) imply (1.26) and (1.27). Since

$$h^{-1}g^{-1} = (gh)^{-1} < I , \tag{5.28}$$

the sequence $\{x_n\}$ converges to zero. By similar arguments, if x_{-1} is defined as $g(x_0)$, the sequence $\{x_n\}$ will still converge. Theorem 20 therefore implies that $g(x_0) = h(x_0)$. Since $h = g$, the convergence for g_1, insert μ into the proper positions of (1.39) and (1.40), and continue the argument of the paragraph containing (1.39) and (1.40). Uniqueness of h is obtained from Theorem 21.

Corollary: Let h and g be two homeomorphisms such that

$$h = P + \mu g^{-1} \quad \text{and} \tag{5.15}$$

$$g = P + \mu h^{-1} .$$

Let $\mu \leq 1$ and $P + \mu I > I$. Then $h = g$.

Theorem 23: Let $\mu \leq 1$ and $P + \mu I > I$. Then a sequence generated by (5.16) will converge if and only if $x_{-1} = h(x_0)$ where h is the homeomorphism of Theorem 22. Furthermore, if it does converge, it will converge to zero.

If $h(x_0) > x_{-1}$, then all of the evenly subscripted elements of the sequence are positive, but all but a finite number of the elements with odd subscripts are negative.

If $h(x_0) < x_{-1}$, then all of the odd subscripted elements are positive and all but a finite number of the even subscripted elements are negative.

Proof: $P + \mu I > I$ and (5.21) imply that $h > I$. If $h(x_0) = x_{-1}$, then $\left\{ x_n \right\} = \left\{ h^{-n}(x_0) \right\}$ and the sequence converges to zero. When $h(x_0) \neq x_{-1}$, then (5.19) and (5.20) may replace (1.33) and (1.34) in order to continue the arguments of the paragraphs containing (1.33) through (1.36).

Theorem 24: Let μ^{-1} be an integer, let P map integers into integer multiples of μ and let $P + \mu I > I$. Then the h satisfying

$$h = P + \mu h^{-1} \tag{5.21}$$

will never map a positive integer into an integer.

Proof: Use the proof presented in the last paragraph of Section I.

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