PROPERTIES OF THE POLYNOMIALS DEFINED BY MORGAN-VOYCE

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1. Introduction

In dealing with electrical ladder networks, A. M. Morgan-Voyce defined a set of polynomials by:

(1) $$b_n(x) = x\, B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

(2) $$B_n(x) = (x+1)\, B_{n-1}(x) + b_{n-1}(x) \qquad (n \geq 1)$$

with,

(3) $$b_0(x) = B_0(x) = 1$$

These polynomials b_n and B_n have a number of very fascinating and interesting properties, and is the subject matter of this article. A few properties of these have been studied by Basin.

From (1) and (2) we see that

(4) $$b_n = B_n - B_{n-1}$$

(5) and, $$x\, B_n = b_{n+1} - b_n$$

Substituting (4) in (1) we have that B_n satisfies the difference equation,

$$B_n(x) = (x+2)\, B_{n-1}(x) - B_{n-2}(x) \qquad (n \geq 2)$$

with

(6) $$B_0(x) = 1, \text{ and } B_1(x) = x + 2$$

From (1) and (2) it can easily be derived that $b_n(x)$ also satisfies the same difference equation, namely,

$$b_n(x) = (x+2)\, b_{n-1}(x) - b_{n-2}(x) \qquad (n \geq 2)$$

with

(7) $$b_0(x) = 1, \text{ and } b_1(x) = x + 1$$

The difference equation (6) may be expressed as the continuant,

(8)
$$B_n(x) = \begin{vmatrix} x+2 & 1 & 0 & . & . & .0 \\ 1 & x+2 & 1 & 0 & . & 0 \\ 0 & 1 & x+2 & 1 & . & \\ . & . & . & . & . & \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 1 \\ 0 & . & . & 0 & 1 & x+2 \end{vmatrix}_n \quad (n \geq 1)$$

and hence we may study the properties of B_n by using those of the continuants. We shall list below only such of those properties of $B_n(x)$ which we will use in studying $b_n(x)$ and in deriving relations between the polynomials $b_n(x)$ and $B_n(x)$:

(9)
$$B_{m+n} = B_m \quad B_n - B_{m-1} \quad B_{n-1}$$

(10)
$$B_{2n} = B_n^2 - B_{n-1}^2$$

(11)
$$B_{2n-1} = B_{n-1} \quad (B_n - B_{n-2})$$

(12)
$$(x + 2) \quad B_{2n-1} = B_n^2 - B_{n-2}^2$$

(13)
$$B_n \quad B_{r-h+1} = B_r \quad B_{n-h+1} - B_{h-2} \quad B_{n-r-1}$$

(14)
$$B_{n-1} \quad B_{n+1} - B_n^2 = -1$$

(15)
$$\frac{d}{dx} \quad B_n(x) = \sum_0^{n-1} (B_r \cdot B_{n-1-r})$$

2. Relations between $b_n(x)$ and $B_n(x)$, and properties of $b_n(x)$:

From (5) and (7),

(16)
$$x \quad B_n = (x + 1) \ b_n - b_{n-1}$$

Also we have,

(17) $$B_{n+1} - B_{n-1} = b_{n+1} + b_n$$

From (4) and (5),

(18) $$b_{n+1} - b_{n-1} = x \quad (B_n + B_{n-1})$$

By successively substituting 0, 1, 2, ... for n in (5) and adding we have,

(19) $$x \sum_{0}^{n} B_r = b_{n+1} - 1$$

Similarly from (4) we may deduce that,

(20) $$\sum_{0}^{n} b_r = B_n$$

Now,

$$b_{m+n} = B_{m+n} - B_{m+n-1} = (B_m B_n - B_{m-1} B_{n-1}) - (B_m B_{n-1} - B_{m-1} B_{n-2})$$

$$= B_m \quad (B_n - B_{n-1}) - (B_{n-1} - B_{n-2}) \quad B_{m-1}$$

Hence,

(21a) $$b_{m+n} = B_m b_n - B_{m-1} b_{n-1}$$

Interchanging m and n we have,

(21b) $$b_{m+n} = b_m B_n - b_{m-1} B_{n-1}$$

Hence,

(22) $$b_m B_n - B_m b_n = b_{m-1} B_{n-1} - B_{m-1} b_{n-1}$$

We will see later that this is a particular case of the more general relationship (29).

Putting $m = n$ in (21),

(23) $$b_{2n} = b_n B_n - b_{n-1} B_{n-1}$$

Putting $m = n+1$ in (21),

(24a) $$b_{2n+1} = b_{n+1}B_n - b_nB_{n-1}$$

(24b) $$= B_{n+1}b_n - B_nb_{n-1}$$

From (7) we have

$$(x + 2)\ b_{2n+1} = b_{2n+2} + b_{2n}$$

$$= b_{n+1}B_{n+1} - b_nB_n + b_nB_n - b_{n-1}B_{n-1}$$

Hence,

(24c) $$(x + 2)\ b_{2n+1} = b_{n+1}B_{n+1} - b_{n-1}B_{n-1}$$

Also from (12),

$$(x + 2)\ B_{2n+1} = B_{n+1}^2 - B_{n-1}^2$$

Hence,

$$(x + 2)(B_{2n+1} - b_{2n+1}) = B_{n+1}(B_{n+1} - b_{n+1}) - B_{n-1}(B_{n-1} - b_{n-1})$$

Hence,

(25) $$(x + 2)\ B_{2n} = B_{n+1}B_n - B_{n-1}B_{n-2}$$

From (23) and (24) we deduce that,

(26) $$b_{2n} - b_{2n-1} = b_n^2 - b_{n-1}^2$$

Subtracting (12) from (25),

$$(x + 2)(B_{2n} - B_{2n-1}) = B_n(B_{n+1} - B_n) - B_{n-2}(B_{n-1} - B_{n-2})$$

Hence,

(27a) $$(x + 2)\ b_{2n} = B_nb_{n+1} - B_{n-2}b_{n-1}$$

(27b) $$= b_nB_{n+1} - b_{n-2}B_{n-1}$$

We will now derive a relationship between the polynomials $b_n(x)$ and $B_n(x)$, corresponding to the relation (13) for B_n:

Consider the expression,

$$b_{n-h+1}B_r - B_{h-2}b_{n-r-1}$$

$$= (B_{n-h+1} - B_{n-h})B_r - (B_{n-r-1} - B_{n-r-2})B_{h-2}$$

$$= (B_{n-h+1}B_r - B_{n-r-1}B_{h-2}) - (B_{n-h}B_r - B_{n-r-2}B_{h-2})$$

$$= B_nB_{r-h+1} - B_{n-1}B_{r-h+1} \qquad \text{from (13)}$$

$$= (B_n - B_{n-1})B_{r-h+1} = b_nB_{r-h+1}$$

Hence,

(28a) $$b_nB_{r-h+1} = b_{n-h+1}B_r - B_{h-2}b_{n-r-1}$$

Similarly,

(28b) $$b_nB_{r-h+1} = B_{n-h+1}b_r - b_{h-2}B_{n-r-1}$$

Hence from (28a) and (28b) we get the relation,

$$B_rb_{n-h+1} - B_{h-2}b_{n-r-1} = b_rB_{n-h+1} - b_{h-2}B_{n-r-1}$$

Changing r to m, h-2 to m-4, and n to m+n+1-r in the above relation,

(29a) $$B_mb_n - B_{m-r}b_{n-r} = b_mB_n - b_{m-r}B_{n-r}$$

Using the relation (4) in (29a) we derive the corresponding relation for $B_n(x)$ as,

(30a) $$B_mB_{n-1} - B_{m-r}B_{n-r-1} = B_nB_{m-1} - B_{n-r}B_{m-r-1}$$

These relations may be written neatly in the form of determinants:

(29b) $$\begin{vmatrix} B_m & B_n \\ b_m & b_n \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{n-r} \\ b_{m-r} & b_{n-r} \end{vmatrix}$$

and

(30b) $$\begin{vmatrix} B_m & B_{m-1} \\ B_n & B_{n-1} \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{m-1-r} \\ B_{n-r} & B_{n-1-r} \end{vmatrix}$$

Now putting $h = 2$, and $n = r+1$ in equation (28) we get,

(31) $$b_rB_r - b_{r+1}B_{r-1} = 1$$

Putting $m = n-1$, and $r = n-1$ in (29b) we get,

(32) $$B_n b_{n-1} - b_n B_{n-1} = 1$$

From (31) and (32) we see that $b_n(x)$ is prime to $b_{n-1}(x)$, $B_n(x)$ and $B_{n-1}(x)$ for integral values of x. Also, for integral values of x, $B_n(x)$ is prime to $B_{n-1}(x)$, $b_n(x)$ and $b_{n+1}(x)$.

By successively substituting 1, 2, 3, ... for n in (10) and adding, we have

$$\sum_1^n B_{2r} = B_n^2 - B_0^2 = B_n^2 - B_0$$

Hence,

(33) $$\sum_0^n B_{2r} = B_n^2$$

Similarly, using (11), (23), (24) and (26) we derive:

(34) $$\sum_0^{n-1} B_{2r+1} = B_n B_{n-1}$$

(35) $$\sum_0^n B_{2r} = b_n B_n$$

(36) $$\sum_0^{n-1} b_{2r+1} = b_n B_{n-1}$$

(37) $$\sum_0^{2n} (-1)^r b_r = b_n^2$$

Let us now find an expression for the derivative of $b_n(x)$:

$$b_n'(x) = B_n' - B_{n-1}' = \sum_0^{n-1} B_r B_{n-1-r} - \sum_0^{n-2} B_r B_{n-2-r}$$

$$= B_{n-1}B_0 + \sum_0^{n-2} B_r(B_{n-r-1} - B_{n-r-2}) = B_{n-1}b_0 + \sum_0^{n-2} B_r b_{n-r-1}$$

Hence,

$$b_n(x) = \sum_{0}^{n-1} B_r b_{n-1-r} \tag{38}$$

3. Explicit polynomial expressions for $B_n(x)$ and $b_n(x)$:

We can establish by induction that,

$$B_n(x) = \sum_{k=0}^{n} (c_n^k x^k) \quad ,$$

where,

$$c_n^k = \binom{n+k+1}{n-k} \quad . \tag{39}$$

Now

$$b_n(x) = B_n(x) - B_{n-1}(x) = \sum_{0}^{n} \left[\binom{n+k+1}{n-k} - \binom{n+k}{n-k-1} \right] x^k \tag{39}$$

$$= \sum_{0}^{n} \binom{n+k}{n-k} x^k \quad .$$

Therefore we have

$$b_n(x) = \sum_{k=0}^{n} (d_n^k x^k)$$

where,

$$d_n^k = \binom{n+k}{n-k} \quad . \tag{40}$$

The equations (39) and (40 are explicit polynomial expressions for b_n and B_n, and show that they are of degree n.

We shall now derive a formula for

$$\int B_n(x) \; dx \quad :$$

From (39),

$$\int B_n(x)\,dx = \sum_0^n \left(c_n^k x^{k+1}/k+1\right) + c$$

Now the coefficient of x^{k+1} for the expression $B_{n+1} - B_{n-1}$ is,

$$c_{n+1}^{k+1} - c_{n-1}^{k+1} = \binom{n+k+3}{n-k} - \binom{n+k+1}{n-k-2} = (n+1)\, c_n^k / (k+1)$$

$$= (n+1)(\text{coefficient of } x^{k+1} \text{ in } \int B_n(x)\,dx\,.)$$

Hence,

(41) $$\int B_n(x)\,dx = \frac{B_{n+1} - B_{n-1}}{n+1} + c \quad .$$

It may also be established that over the interval (-4, 0), $B_n(x)$ and $b_n(x)$ are orthogonal polynomials with respect to the weight functions $\sqrt{4 - (x+2)^2}$ and $\sqrt{(x+4)/-x}$ respectively.

It may also be seen from (6) that,

(42a) $$B_n(x) = S_n(x+2)$$

and hence,

(42b) $$b_n(x) = S_n(x+2) - S_{n-1}(x+2) \quad ,$$

where $S_n(x)$ is the Chebyshev polynomial.

4. Conclusions:

The article deals with the properties of a set of polynomials $b_n(x)$ and $B_n(x)$ defined by (1), (2) and (3). Even though they are related to the Chebyshev polynomials, the author believes that $B_n(x)$ and $b_n(x)$ are of use in the study of ladder networks and hence deserve a study of this nature.

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ACKNOWLEDGEMENTS

This work was supported by the National Research Council of Canada, under Grant No. A-2875. The author would like to thank the reviewer and Professor V. E. Hoggatt for their useful suggestions and comments.

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