POWERS OF THE GOLDEN SECTION

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The Golden Section is the positive root k of the equation $x^2 + x = 1$,

$$k = (\sqrt{5} - 1)/2 = 0.618034.$$

The negative root, $(-\sqrt{5} - 1)/2$, is $-1/k$, the negative reciprocal of k. From the above equation,

$$k^n = k^{n+1} + k^{n+2} = k^{n-2} - k^{n-1},$$

That is, any power of k is the sum of the next two higher powers or the difference between the next two lower powers.

If the powers of k are listed serially as in the box tabulation with the accompanying diagram, the ascending and descending ratios of the successive powers are k and $1/k$ respectively.

Each power of k can be expressed in terms of its first power and a Fibonacci number as indicated in the right column of the box tabulation. Starting with the successive powers k^0 and k^1, this column can be completed by repeated application of the k^n formula. It is evident from this tabulation and the k^n formula that the powers of k form a Fibonacci series which can be separated into two component Fibonacci series.

Continuing further, all power of k can be expressed in terms of any other power and a number as shown on the accompanying transformation tables. The box tabulation on the diagram can be used to determine the values given in the transformation table. For example, the value $k = \frac{1}{8} k^6 - \frac{5}{8}$ is obtained from $k^6 = 5 - 8k$. This value for k, coupled with the value $k^0 = 0 + \frac{8}{8}$, can be used to determine all of the values listed in the vertical k^6 column with the aid of the k^n formulas.

It is interesting to note the recurrence of the Fibonacci sequence in the numerators in the vertical columns and in the denominators in the horizontal columns. Both the Fibonacci and Lucas series appear

in the k^0 vertical column. Complex expressions involving various powers of k can be very much simplified by reference to these tables.

REFERENCE

Robert S. Beard, "The Golden Section and Fibonacci Numbers", Scripta Mathematica, Vol. 16, Mar. - June, 1950 pp. 116-119.

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(Tables and Chart are on pages 165, 166 and 167.)

In general we may transform equation (1) by writing

(3) $$u_k = F(V_k) \quad .$$

Suppose that furthermore we require that F satisfy the functional equation

(4) $$F^2(s) - bF(s) + b = F(2s) \quad .$$

Then our equation becomes $F(V_{k+1}) = F(2V_k)$, a solution of which is given by $V_k = A2^k$. Hence we have

(5) $$u_k = F(A2^k) \quad .$$

We now consider the functional equation (4). Let

$$F(s) - \frac{b}{2} = 2L(s) \quad .$$

Then equation (4) becomes

(6) $$L^2(s) = \frac{1}{2} \left[(\frac{b^2}{8} - \frac{b}{4}) + L(2s) \right] \quad .$$

TRANSFORMATION OF POWERS OF K. (Equivalents in Other Powers.)

Powers of K.	K^0	K^1	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}	R.S.B.	Powers of K.
10	$\frac{55\sqrt5}{2}+\frac{123}{2}$	$55K+34$	$55K^2-21$	$\frac{55K^3}{2}+\frac{13}{2}$	$\frac{55K^4}{3}-\frac{8}{3}$	$\frac{55K^5}{5}+\frac{5}{5}$	$\frac{55K^6}{8}-\frac{3}{8}$	$\frac{55K^7}{13}+\frac{2}{13}$	$\frac{55K^8}{21}-\frac{1}{21}$	$\frac{55K^9}{34}+\frac{1}{34}$	$\frac{55K^{10}}{55}+0$	$\frac{55K^{10}}{55}+0$	10
9	$\frac{34\sqrt5}{2}+\frac{76}{2}$	$34K+21$	$34K^2-13$	$\frac{34K^3}{2}+\frac{8}{2}$	$\frac{34K^4}{3}-\frac{5}{3}$	$\frac{34K^5}{5}+\frac{3}{5}$	$\frac{34K^6}{8}-\frac{2}{8}$	$\frac{34K^7}{13}+\frac{1}{13}$	$\frac{34K^8}{21}-\frac{1}{21}$	$\frac{34K^9}{34}+0$	$\frac{34K^{10}}{55}-\frac{1}{55}$	$\frac{34K^{10}}{55}-\frac{1}{55}$	9
8	$\frac{21\sqrt5}{2}+\frac{47}{2}$	$21K+13$	$21K^2-8$	$\frac{21K^3}{2}+\frac{5}{2}$	$\frac{21K^4}{3}-\frac{3}{3}$	$\frac{21K^5}{5}+\frac{2}{5}$	$\frac{21K^6}{8}-\frac{1}{8}$	$\frac{21K^7}{13}+\frac{1}{13}$	$\frac{21K^8}{21}+0$	$\frac{21K^9}{34}+\frac{1}{34}$	$\frac{21K^{10}}{55}+\frac{1}{55}$	$\frac{21K^{10}}{55}+\frac{1}{55}$	8
7	$\frac{13\sqrt5}{2}+\frac{29}{2}$	$13K+8$	$13K^2-5$	$\frac{13K^3}{2}+\frac{3}{2}$	$\frac{13K^4}{3}-\frac{2}{3}$	$\frac{13K^5}{5}+\frac{1}{5}$	$\frac{13K^6}{8}-\frac{1}{8}$	$\frac{13K^7}{13}+0$	$\frac{13K^8}{21}-\frac{1}{21}$	$\frac{13K^9}{34}-\frac{1}{34}$	$\frac{13K^{10}}{55}-\frac{2}{55}$	$\frac{13K^{10}}{55}-\frac{2}{55}$	7
6	$\frac{8\sqrt5}{2}+\frac{18}{2}$	$8K+5$	$8K^2-3$	$\frac{8K^3}{2}+\frac{2}{2}$	$\frac{8K^4}{3}-\frac{1}{3}$	$\frac{8K^5}{5}+\frac{1}{5}$	$\frac{8K^6}{8}+0$	$\frac{8K^7}{13}+\frac{1}{13}$	$\frac{8K^8}{21}+\frac{1}{21}$	$\frac{8K^9}{34}+\frac{2}{34}$	$\frac{8K^{10}}{55}+\frac{3}{55}$	$\frac{8K^{10}}{55}+\frac{3}{55}$	6
5	$\frac{5\sqrt5}{2}+\frac{11}{2}$	$5K+3$	$5K^2-2$	$\frac{5K^3}{2}+\frac{1}{2}$	$\frac{5K^4}{3}-\frac{1}{3}$	$\frac{5K^5}{5}+0$	$\frac{5K^6}{8}-\frac{1}{8}$	$\frac{5K^7}{13}-\frac{1}{13}$	$\frac{5K^8}{21}-\frac{2}{21}$	$\frac{5K^9}{34}-\frac{3}{34}$	$\frac{5K^{10}}{55}-\frac{5}{55}$	$\frac{5K^{10}}{55}-\frac{5}{55}$	5
4	$\frac{3\sqrt5}{2}+\frac{7}{2}$	$3K+2$	$3K^2-1$	$\frac{3K^3}{2}+\frac{1}{2}$	$\frac{3K^4}{3}+0$	$\frac{3K^5}{5}+\frac{1}{5}$	$\frac{3K^6}{8}+\frac{1}{8}$	$\frac{3K^7}{13}+\frac{2}{13}$	$\frac{3K^8}{21}+\frac{3}{21}$	$\frac{3K^9}{34}+\frac{5}{34}$	$\frac{3K^{10}}{55}+\frac{8}{55}$	$\frac{3K^{10}}{55}+\frac{8}{55}$	4
3	$\frac{2\sqrt5}{2}+\frac{4}{2}$	$2K+1$	$2K^2-1$	$\frac{2K^3}{2}+0$	$\frac{2K^4}{3}-\frac{1}{3}$	$\frac{2K^5}{5}-\frac{1}{5}$	$\frac{2K^6}{8}-\frac{2}{8}$	$\frac{2K^7}{13}-\frac{3}{13}$	$\frac{2K^8}{21}-\frac{5}{21}$	$\frac{2K^9}{34}-\frac{8}{34}$	$\frac{2K^{10}}{55}-\frac{13}{55}$	$\frac{2K^{10}}{55}-\frac{13}{55}$	3
2	$\frac{\sqrt5}{2}+\frac{3}{2}$	$K+1$	K^2+0	$\frac{K^3}{2}+\frac{1}{2}$	$\frac{K^4}{3}+\frac{1}{3}$	$\frac{K^5}{5}+\frac{2}{5}$	$\frac{K^6}{8}+\frac{3}{8}$	$\frac{K^7}{13}+\frac{5}{13}$	$\frac{K^8}{21}+\frac{8}{21}$	$\frac{K^9}{34}+\frac{13}{34}$	$\frac{K^{10}}{55}+\frac{21}{55}$	$\frac{K^{10}}{55}+\frac{21}{55}$	2
1	$\frac{\sqrt5}{2}+\frac{1}{2}$	$K+0$	K^2-1	$\frac{K^3}{2}-\frac{1}{2}$	$\frac{K^4}{3}-\frac{2}{3}$	$\frac{K^5}{5}-\frac{3}{5}$	$\frac{K^6}{8}-\frac{5}{8}$	$\frac{K^7}{13}-\frac{8}{13}$	$\frac{K^8}{21}-\frac{13}{21}$	$\frac{K^9}{34}-\frac{21}{34}$	$\frac{K^{10}}{55}-\frac{34}{55}$	$\frac{K^{10}}{55}-\frac{34}{55}$	1
0	$0+1$	$0+1$	$0+1$	$0+\frac{2}{2}$	$0+\frac{3}{3}$	$0+\frac{5}{5}$	$0+\frac{8}{8}$	$0+\frac{13}{13}$	$0+\frac{21}{21}$	$0+\frac{34}{34}$	$0+\frac{55}{55}$	$0+\frac{55}{55}$	0
-1	$\frac{\sqrt5}{2}-\frac{1}{2}$	$K-1$	K^2-2	$\frac{K^3}{2}-\frac{3}{2}$	$\frac{K^4}{3}-\frac{5}{3}$	$\frac{K^5}{5}-\frac{8}{5}$	$\frac{K^6}{8}-\frac{13}{8}$	$\frac{K^7}{13}-\frac{21}{13}$	$\frac{K^8}{21}-\frac{34}{21}$	$\frac{K^9}{34}-\frac{55}{34}$	$\frac{K^{10}}{55}-\frac{89}{55}$	$\frac{K^{10}}{55}-\frac{89}{55}$	-1
-2	$-\frac{\sqrt5}{2}+\frac{3}{2}$	$-K+2$	$-K^2+3$	$-\frac{K^3}{2}+\frac{5}{2}$	$-\frac{K^4}{3}+\frac{8}{3}$	$-\frac{K^5}{5}+\frac{13}{5}$	$-\frac{K^6}{8}+\frac{21}{8}$	$-\frac{K^7}{13}+\frac{34}{13}$	$-\frac{K^8}{21}+\frac{55}{21}$	$-\frac{K^9}{34}+\frac{89}{34}$	$-\frac{K^{10}}{55}+\frac{144}{55}$	$-\frac{K^{10}}{55}+\frac{144}{55}$	-2
-3	$\frac{2\sqrt5}{2}-\frac{4}{2}$	$2K-3$	$2K^2-5$	$\frac{2K^3}{2}-\frac{8}{2}$	$\frac{2K^4}{3}-\frac{13}{3}$	$\frac{2K^5}{5}-\frac{21}{5}$	$\frac{2K^6}{8}-\frac{34}{8}$	$\frac{2K^7}{13}-\frac{55}{13}$	$\frac{2K^8}{21}-\frac{89}{21}$	$\frac{2K^9}{34}-\frac{144}{34}$	$\frac{2K^{10}}{55}-\frac{233}{55}$	$\frac{2K^{10}}{55}-\frac{233}{55}$	-3
-4	$-\frac{3\sqrt5}{2}+\frac{7}{2}$	$-3K+5$	$-3K^2+8$	$-\frac{3K^3}{2}+\frac{13}{2}$	$-\frac{3K^4}{3}+\frac{21}{3}$	$-\frac{3K^5}{5}+\frac{34}{5}$	$-\frac{3K^6}{8}+\frac{55}{8}$	$-\frac{3K^7}{13}+\frac{89}{13}$	$-\frac{3K^8}{21}+\frac{144}{21}$	$-\frac{3K^9}{34}+\frac{233}{34}$	$-\frac{3K^{10}}{55}+\frac{377}{55}$	$-\frac{3K^{10}}{55}+\frac{377}{55}$	-4
-5	$\frac{5\sqrt5}{2}-\frac{11}{2}$	$5K-8$	$5K^2-13$	$\frac{5K^3}{2}-\frac{21}{2}$	$\frac{5K^4}{3}-\frac{34}{3}$	$\frac{5K^5}{5}-\frac{55}{5}$	$\frac{5K^6}{8}-\frac{89}{8}$	$\frac{5K^7}{13}-\frac{144}{13}$	$\frac{5K^8}{21}-\frac{233}{21}$	$\frac{5K^9}{34}-\frac{377}{34}$	$\frac{5K^{10}}{55}-\frac{610}{55}$	$\frac{5K^{10}}{55}-\frac{610}{55}$	-5
-6	$-\frac{8\sqrt5}{2}+\frac{18}{2}$	$-8K+13$	$-8K^2+21$	$-\frac{8K^3}{2}+\frac{34}{2}$	$-\frac{8K^4}{3}+\frac{55}{3}$	$-\frac{8K^5}{5}+\frac{89}{5}$	$-\frac{8K^6}{8}+\frac{144}{8}$	$-\frac{8K^7}{13}+\frac{233}{13}$	$-\frac{8K^8}{21}+\frac{377}{21}$	$-\frac{8K^9}{34}+\frac{610}{34}$	$-\frac{8K^{10}}{55}+\frac{987}{55}$	$-\frac{8K^{10}}{55}+\frac{987}{55}$	-6
-7	$\frac{13\sqrt5}{2}-\frac{29}{2}$	$13K-21$	$13K^2-34$	$\frac{13K^3}{2}-\frac{55}{2}$	$\frac{13K^4}{3}-\frac{89}{3}$	$\frac{13K^5}{5}-\frac{144}{5}$	$\frac{13K^6}{8}-\frac{233}{8}$	$\frac{13K^7}{13}-\frac{377}{13}$	$\frac{13K^8}{21}-\frac{610}{21}$	$\frac{13K^9}{34}-\frac{987}{34}$	$\frac{13K^{10}}{55}-\frac{1597}{55}$	$\frac{13K^{10}}{55}-\frac{1597}{55}$	-7
-8	$-\frac{21\sqrt5}{2}+\frac{47}{2}$	$-21K+34$	$-21K^2+55$	$-\frac{21K^3}{2}+\frac{89}{2}$	$-\frac{21K^4}{3}+\frac{144}{3}$	$-\frac{21K^5}{5}+\frac{233}{5}$	$-\frac{21K^6}{8}+\frac{377}{8}$	$-\frac{21K^7}{13}+\frac{610}{13}$	$-\frac{21K^8}{21}+\frac{987}{21}$	$-\frac{21K^9}{34}+\frac{1597}{34}$	$-\frac{21K^{10}}{55}+\frac{2584}{55}$	$-\frac{21K^{10}}{55}+\frac{2584}{55}$	-8
-9	$\frac{34\sqrt5}{2}-\frac{76}{2}$	$34K-55$	$34K^2-89$	$\frac{34K^3}{2}-\frac{144}{2}$	$\frac{34K^4}{3}-\frac{233}{3}$	$\frac{34K^5}{5}-\frac{377}{5}$	$\frac{34K^6}{8}-\frac{610}{8}$	$\frac{34K^7}{13}-\frac{987}{13}$	$\frac{34K^8}{21}-\frac{1597}{21}$	$\frac{34K^9}{34}-\frac{2584}{34}$	$\frac{34K^{10}}{55}-\frac{4181}{55}$	$\frac{34K^{10}}{55}-\frac{4181}{55}$	-9
-10	$-\frac{55\sqrt5}{2}+\frac{123}{2}$	$-55K+89$	$-55K^2+144$	$-\frac{55K^3}{2}+\frac{233}{2}$	$-\frac{55K^4}{3}+\frac{377}{3}$	$-\frac{55K^5}{5}+\frac{610}{5}$	$-\frac{55K^6}{8}+\frac{987}{8}$	$-\frac{55K^7}{13}+\frac{1597}{13}$	$-\frac{55K^8}{21}+\frac{2584}{21}$	$-\frac{55K^9}{34}+\frac{4181}{34}$	$-\frac{55K^{10}}{55}+\frac{6765}{55}$	$-\frac{55K^{10}}{55}+\frac{6765}{55}$	-10

TRANSFORMATION OF POWERS OF K. (Equivalents in Other Powers.)

RSB.

Powers of K	K^0	K	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}	Powers of K
-10	122.991870	$55k+89$	$-55k^2+144$	$\frac{55}{2}k^3+\frac{233}{2}$	$-\frac{55}{3}k^4+\frac{377}{3}$	$\frac{55}{5}k^5+\frac{610}{5}$	$-\frac{55}{8}k^6+\frac{987}{8}$	$\frac{55}{13}k^7+\frac{1597}{13}$	$-\frac{55}{21}k^8+\frac{2584}{21}$	$\frac{55}{34}k^9+\frac{4181}{34}$	$-\frac{55}{55}k^{10}+\frac{6765}{55}$	-10
-9	76.013156	$34k+55$	$-34k^2+89$	$\frac{34}{2}k^3+\frac{144}{2}$	$-\frac{34}{3}k^4+\frac{233}{3}$	$\frac{34}{5}k^5+\frac{377}{5}$	$-\frac{34}{8}k^6+\frac{610}{8}$	$\frac{34}{13}k^7+\frac{987}{13}$	$-\frac{34}{21}k^8+\frac{1597}{21}$	$\frac{34}{34}k^9+\frac{2584}{34}$	$-\frac{34}{55}k^{10}+\frac{4181}{55}$	-9
-8	46.978714	$21k+34$	$-21k^2+55$	$\frac{21}{2}k^3+\frac{89}{2}$	$-\frac{21}{3}k^4+\frac{144}{3}$	$\frac{21}{5}k^5+\frac{233}{5}$	$-\frac{21}{8}k^6+\frac{377}{8}$	$\frac{21}{13}k^7+\frac{610}{13}$	$-\frac{21}{21}k^8+\frac{987}{21}$	$\frac{21}{34}k^9+\frac{1597}{34}$	$-\frac{21}{55}k^{10}+\frac{2584}{55}$	-8
-7	29.034442	$13k+21$	$-13k^2+34$	$\frac{13}{2}k^3+\frac{55}{2}$	$-\frac{13}{3}k^4+\frac{89}{3}$	$\frac{13}{5}k^5+\frac{144}{5}$	$-\frac{13}{8}k^6+\frac{233}{8}$	$\frac{13}{13}k^7+\frac{377}{13}$	$-\frac{13}{21}k^8+\frac{610}{21}$	$\frac{13}{34}k^9+\frac{987}{34}$	$-\frac{13}{55}k^{10}+\frac{1597}{55}$	-7
-6	17.944272	$8k+13$	$-8k^2+21$	$\frac{8}{2}k^3+\frac{34}{2}$	$-\frac{8}{3}k^4+\frac{55}{3}$	$\frac{8}{5}k^5+\frac{89}{5}$	$-\frac{8}{8}k^6+\frac{144}{8}$	$\frac{8}{13}k^7+\frac{233}{13}$	$-\frac{8}{21}k^8+\frac{377}{21}$	$\frac{8}{34}k^9+\frac{610}{34}$	$-\frac{8}{55}k^{10}+\frac{987}{55}$	-6
-5	11.090170	$5k+8$	$-5k^2+13$	$\frac{5}{2}k^3+\frac{21}{2}$	$-\frac{5}{3}k^4+\frac{34}{3}$	$\frac{5}{5}k^5+\frac{55}{5}$	$-\frac{5}{8}k^6+\frac{89}{8}$	$\frac{5}{13}k^7+\frac{144}{13}$	$-\frac{5}{21}k^8+\frac{233}{21}$	$\frac{5}{34}k^9+\frac{377}{34}$	$-\frac{5}{55}k^{10}+\frac{610}{55}$	-5
-4	6.854102	$3k+5$	$-3k^2+8$	$\frac{3}{2}k^3+\frac{13}{2}$	$-\frac{3}{3}k^4+\frac{21}{3}$	$\frac{3}{5}k^5+\frac{34}{5}$	$-\frac{3}{8}k^6+\frac{55}{8}$	$\frac{3}{13}k^7+\frac{89}{13}$	$-\frac{3}{21}k^8+\frac{144}{21}$	$\frac{3}{34}k^9+\frac{233}{34}$	$-\frac{3}{55}k^{10}+\frac{377}{55}$	-4
-3	4.236068	$2k+3$	$-2k^2+5$	$\frac{2}{2}k^3+\frac{8}{2}$	$-\frac{2}{3}k^4+\frac{13}{3}$	$\frac{2}{5}k^5+\frac{21}{5}$	$-\frac{2}{8}k^6+\frac{34}{8}$	$\frac{2}{13}k^7+\frac{55}{13}$	$-\frac{2}{21}k^8+\frac{89}{21}$	$\frac{2}{34}k^9+\frac{144}{34}$	$-\frac{2}{55}k^{10}+\frac{233}{55}$	-3
-2	2.618034	$k+2$	$-k^2+3$	$\frac{1}{2}k^3+\frac{5}{2}$	$-\frac{1}{3}k^4+\frac{8}{3}$	$\frac{1}{5}k^5+\frac{13}{5}$	$-\frac{1}{8}k^6+\frac{21}{8}$	$\frac{1}{13}k^7+\frac{34}{13}$	$-\frac{1}{21}k^8+\frac{55}{21}$	$\frac{1}{34}k^9+\frac{89}{34}$	$-\frac{1}{55}k^{10}+\frac{144}{55}$	-2
-1	1.618034	$k+1$	$-k^2+2$	$\frac{1}{2}k^3+\frac{3}{2}$	$-\frac{1}{3}k^4+\frac{5}{3}$	$\frac{1}{5}k^5+\frac{8}{5}$	$-\frac{1}{8}k^6+\frac{13}{8}$	$\frac{1}{13}k^7+\frac{21}{13}$	$-\frac{1}{21}k^8+\frac{34}{21}$	$\frac{1}{34}k^9+\frac{55}{34}$	$-\frac{1}{55}k^{10}+\frac{89}{55}$	-1
0	1.000000	$0+1$	$0+1$	$0+\frac{2}{2}$	$0+\frac{3}{3}$	$0+\frac{5}{5}$	$0+\frac{8}{8}$	$0+\frac{13}{13}$	$0+\frac{21}{21}$	$0+\frac{34}{34}$	$0+\frac{55}{55}$	0
1	0.618034	$k+0$	$-k^2+1$	$\frac{1}{2}k^3+\frac{1}{2}$	$-\frac{1}{3}k^4+\frac{2}{3}$	$\frac{1}{5}k^5+\frac{3}{5}$	$-\frac{1}{8}k^6+\frac{5}{8}$	$\frac{1}{13}k^7+\frac{8}{13}$	$-\frac{1}{21}k^8+\frac{13}{21}$	$\frac{1}{34}k^9+\frac{21}{34}$	$-\frac{1}{55}k^{10}+\frac{34}{55}$	1
2	0.381966	$-k+1$	k^2+0	$-\frac{1}{2}k^3+\frac{1}{2}$	$\frac{1}{3}k^4+\frac{1}{3}$	$-\frac{1}{5}k^5+\frac{2}{5}$	$\frac{1}{8}k^6+\frac{3}{8}$	$-\frac{1}{13}k^7+\frac{5}{13}$	$\frac{1}{21}k^8+\frac{8}{21}$	$-\frac{1}{34}k^9+\frac{13}{34}$	$\frac{1}{55}k^{10}+\frac{21}{55}$	2
3	0.236068	$2k-1$	$-2k^2+1$	$\frac{2}{2}k^3+0$	$-\frac{2}{3}k^4+\frac{1}{3}$	$\frac{2}{5}k^5+\frac{1}{5}$	$-\frac{2}{8}k^6+\frac{2}{8}$	$\frac{2}{13}k^7+\frac{3}{13}$	$-\frac{2}{21}k^8+\frac{5}{21}$	$\frac{2}{34}k^9+\frac{8}{34}$	$-\frac{2}{55}k^{10}+\frac{13}{55}$	3
4	0.145898	$-3k+2$	$3k^2-1$	$-\frac{3}{2}k^3+\frac{1}{2}$	$\frac{3}{3}k^4+0$	$-\frac{3}{5}k^5+\frac{1}{5}$	$\frac{3}{8}k^6+\frac{1}{8}$	$-\frac{3}{13}k^7+\frac{2}{13}$	$\frac{3}{21}k^8+\frac{3}{21}$	$-\frac{3}{34}k^9+\frac{5}{34}$	$\frac{3}{55}k^{10}+\frac{8}{55}$	4
5	0.090170	$5k-3$	$-5k^2+2$	$\frac{5}{2}k^3-\frac{1}{2}$	$-\frac{5}{3}k^4+\frac{1}{3}$	$\frac{5}{5}k^5+0$	$-\frac{5}{8}k^6+\frac{1}{8}$	$\frac{5}{13}k^7+\frac{1}{13}$	$-\frac{5}{21}k^8+\frac{2}{21}$	$\frac{5}{34}k^9+\frac{3}{34}$	$-\frac{5}{55}k^{10}+\frac{5}{55}$	5
6	0.055728	$-8k+5$	$8k^2-3$	$-\frac{8}{2}k^3+\frac{2}{2}$	$\frac{8}{3}k^4-\frac{1}{3}$	$-\frac{8}{5}k^5+\frac{1}{5}$	$\frac{8}{8}k^6+0$	$-\frac{8}{13}k^7+\frac{1}{13}$	$\frac{8}{21}k^8+\frac{1}{21}$	$-\frac{8}{34}k^9+\frac{2}{34}$	$\frac{8}{55}k^{10}+\frac{3}{55}$	6
7	0.034442	$13k-8$	$-13k^2+5$	$\frac{13}{2}k^3-\frac{3}{2}$	$-\frac{13}{3}k^4+\frac{2}{3}$	$\frac{13}{5}k^5-\frac{1}{5}$	$-\frac{13}{8}k^6+\frac{1}{8}$	$\frac{13}{13}k^7+0$	$-\frac{13}{21}k^8+\frac{1}{21}$	$\frac{13}{34}k^9+\frac{1}{34}$	$-\frac{13}{55}k^{10}+\frac{2}{55}$	7
8	0.021286	$-21k+13$	$21k^2-8$	$-\frac{21}{2}k^3+\frac{5}{2}$	$\frac{21}{3}k^4-\frac{3}{3}$	$-\frac{21}{5}k^5+\frac{2}{5}$	$\frac{21}{8}k^6-\frac{1}{8}$	$-\frac{21}{13}k^7+\frac{1}{13}$	$\frac{21}{21}k^8+0$	$-\frac{21}{34}k^9+\frac{1}{34}$	$\frac{21}{55}k^{10}-\frac{1}{55}$	8
9	0.013156	$34k-21$	$-34k^2+13$	$\frac{34}{2}k^3-\frac{8}{2}$	$-\frac{34}{3}k^4+\frac{5}{3}$	$\frac{34}{5}k^5-\frac{3}{5}$	$-\frac{34}{8}k^6+\frac{2}{8}$	$\frac{34}{13}k^7-\frac{1}{13}$	$-\frac{34}{21}k^8+\frac{1}{21}$	$\frac{34}{34}k^9+0$	$-\frac{34}{55}k^{10}+\frac{1}{55}$	9
10	0.008130	$-55k+34$	$55k^2-21$	$-\frac{55}{2}k^3+\frac{13}{2}$	$\frac{55}{3}k^4-\frac{8}{3}$	$-\frac{55}{5}k^5+\frac{5}{5}$	$\frac{55}{8}k^6-\frac{3}{8}$	$-\frac{55}{13}k^7+\frac{2}{13}$	$\frac{55}{21}k^8-\frac{1}{21}$	$-\frac{55}{34}k^9+\frac{1}{34}$	$\frac{55}{55}k^{10}+0$	10

Powers of the Golden Section Ratio, K^n

$$k = \tfrac{1}{2}\sqrt{5} - \tfrac{1}{2} = 0.618034 \qquad K^n = K^{n+1} + K^{n+2} \qquad K^n = K + K^{n-2} = K - K^{n-1}$$

Each power of k is the sum of the next two higher powers or the difference between the next two lower powers. In the right column of the box tabulation, each power of k is expressed in terms of its first power and an integer. In these expressions both the integers and the coefficients of k form Fibonacci series.

n.	Values of K^n	
∞	0.000	—
8	0.021	13-21k
7	0.034	-8+13k
6	0.056	5-8k
5	0.090	-3+5k
4	0.146	2-3k
3	0.236	-1+2k
2	0.382	1-k
1	0.618	0+k
0	1.000	1+0
-1	1.618	1+k
-2	2.618	2+k
-3	4.236	3+2k
-4	6.854	5+3k
-5	11.090	8+5k
-6	17.944	13+8k
-7	29.034	21+13k
-8	46.978	34+21k

Powers of k.

Values of n.

Values of K^n, integers and k terms in right column.

Integers in right column.

K terms in right column.

K^n