TERMINAL DIGIT COINCIDENCES BETWEEN FIBONACCI NUMBERS AND THEIR INDICES

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In [1] the Editor of this journal proposed essentially the following question: What Fibonacci numbers of index less than 10,000 have terminal digits coincident with the index? This note answers that question by supplying a computer-generated table of such coincidences.

The computer program was written in the ALGOL 60 international algorithmic language [2] and is given in the Appendix. For those readers not familiar with the ALGOL language, the operations performed by the program are basically as follows: With starting values given, all Fibonacci numbers with indices 1 to 9999 are computed modulo 10,000. Fibonacci numbers F_1 through F_9 are then reduced mod 10 and compared with their respective indices, F_{10} through F_{99} are reduced mod 100 and likewise compared, and similar reductions and comparisons are performed for F_{100} through F_{999} and for F_{1000} through F_{9999}. If the comparison yields an answer YES, the index for which this occurs is marked with an asterist. A 100 x 100 table, shown listed as tables I and II, is then printed out with row coordinates in hundreds and column coordinates in units, and with asterisks in the locations where coincidences occur. Hence, an asterisk in row zero column 61 indicates that F_{61} has 61 as its last two digits, and similarly an asterisk in row 4 column 85 indicates that F_{485} has 485 as its last three digits. Note the regularity of patterns in Tables I and II; these might prove to be an interesting subject for further investigation.

A digest of the results reported herein is given in Table III. These results were checked through F_{505} by inspecting a table [3] of Fibonacci numbers and by running several versions of the basic program.

REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Numbers with a Calculator," Fibonacci Quarterly, Vol. 2, No. 2 (April 1964), p. 138.

2. P. Naur et al, "Revised Report on the Algorithmic Language ALGOL 60", Communications of the ACM, Vol. 6, No. 1 (January 1963), pp. 1-17.

3. S. L. Basin and V. E. Hoggatt, Jr., "The First 571 Fibonacci Numbers", Recreational Mathematics, No. 11 (October 1962), pp. 19-30. (Continued on page 153.)

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$$u_k - b = u_{k-1}u_{k-2} \cdots u_{k-(k-1)}\left[u_{k-(k-1)} - b\right] \quad .$$

Hence

$$u_{k_1} = (u_0 - b) \prod_{i=0}^{i=k_1-1} u_i + b \quad .$$

Choose $k_2 < k_1$ without loss of generality and the conclusion is apparent.

We now consider equation (1) in several special cases. If $b = 0$ the equation is easily solved but in this case the theorem holds trivially. Let $b = 2$. Then we have

$$u_{k+1} = u_k^2 - 2u_k + 2 \quad ,$$

which can be written in the form

$$u_{k+1} - 1 = (u_k - 1)^2 \quad .$$

The solution of this equation is clearly

$$u_k = A^{2^k} + 1 \quad . \tag{2}$$

Hence the sequence with elements $A^{2^k} + 1$, for integral A, is relatively prime except possibly for the common divisor 2. The exception is obviously removed when A is an even integer. When $A = 2$, we have the Fermat numbers mentioned previously.

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APPENDIX

ALGOL-60 Program for Producing Fibonacci Terminal Digit Concidences

```
begin
     procedure hollerith (name, representation);
     real name;
     string representation;
          name: = representation;
     integer procedure mod (x, m);
     value x, m;
     integer x, m;
          mod: = x - m × (x ÷ m);
     integer i, j, k;
     real blank, star;
     integer array F[-1:9999] ;
     real array cell[0:9999] ;
     hollerith (blank, ' ');
     hollerith (star, '*');
     F[-1] : = 1;
     F[0] : = 0;
     cell[0] : = star;
     for k: = 1 step 1 until 4 do
     begin
          integer finish, modulus;
          modulus: = 10 ↑ k;
          finish: = modulus - 1;
          for i: = modulus / 10 step 1 until finish do
          begin
               F[i]    : = mod (F[i-1] + F[i-2] , 10000);
               cell[i] : = if mod (F[i] , modulus) = i
                           then star
                           else blank;
          end calculations for k-th order of magnitude;
     end setup of coincidence table;
     write (for i: = 0 step 1 until 99 do
            for j: = 0 step 1 until 99 do cell[100×i + j] );
end Fibonacci terminal digit coincidence program;
```

TABLE I

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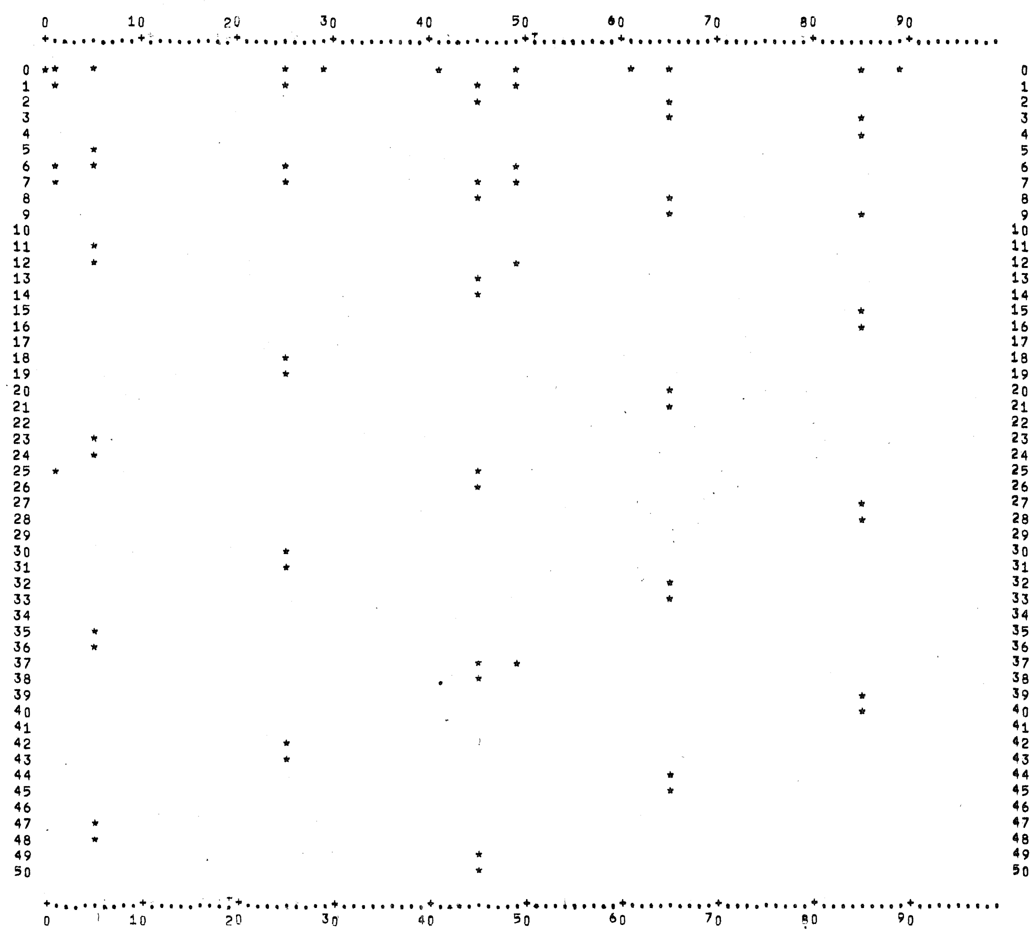

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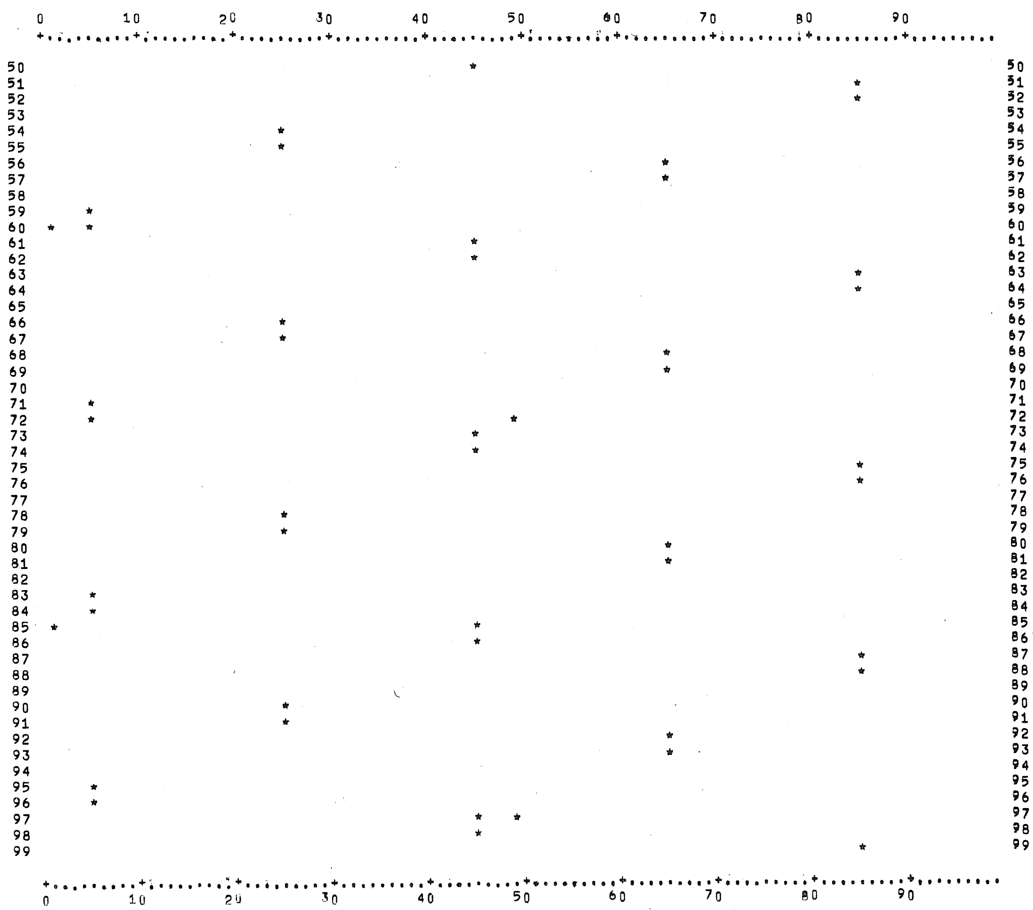

TABLE III

INDICES COINCIDENT WITH FIBONACCI NUMBER TERMINAL DIGITS

1, 5,
25, 29,
41, 49,
61, 65,
85, 89
101, 125, 145, 149,
245, 265,
365, 385,
485, 505,
601, 605, 625, 649,
701, 725, 745, 749,
845, 865,
965, 985,

1105,		1205,	1249,	1345,		1445,	1585,	1685,	1825,	1925,	2065,	21
2305,		2405,	2501,	2545,		2645,	2785,	2885,	3025,	3125,	3265,	33
3505,		3605,		3745,	3749,	3845,	3985,	4085,	4225,	4325,	4465,	45
4705,		4805,		4945,		5045,	5185,	5285,	5425,	5525,	5665,	57
5905,	6001,	6005,		6145,		6245,	6385,	6485,	6625,	6725,	6865,	69
7105,		7205,	7249,	7345,		7445,	7585,	7685,	7825,	7925,	8065,	81
8305,		8405,	8501,	8545,		8645,	8785,	8885,	9025,	9125,	9265,	93
9505,		9605,		9745,	9749,	9845,	9985,	others exceed 10000.				

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Also solved by Douglas Lind and Donald Howells.