A GENERALIZED LANGFORD PROBLEM

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Let $n > 1$ be an integer and consider the integers 1, 2, 3, ..., n. The sequence $a_1, a_2, a_3, \ldots, a_{2n}$ is said to be a perfect sequence for n if each of the integers 1, 2, 3, ..., n occurs in the sequence exactly twice and the integer i is separated in the sequence by exactly i entries. For example, 1 7 1 2 6 4 2 5 3 7 4 6 3 5 is a perfect sequence for 7. C. D. Langford [2] posed the problem of determining all n having a perfect sequence. It was shown by C. J. Priday 3 and Roy O. Davies 1 that n has a perfect sequence if, and only if, n is of the form $4m - 1$ or $4m$. For $n = 3$, 3 1 2 1 3 2 is the only perfect sequence except for the same sequence in reverse order and for $n = 4$, 4 1 3 1 2 4 3 2 is the only perfect sequence except for the same sequence in reverse order. According to Davies there are 25 perfect sequences for 7. He stated the problem, as yet unsolved, of finding a function giving the number of perfect sequences for n of the form $4m - 1$ or $4m$.

In this note we define a generalized perfect s-sequence for the integer $n > 1$ to be a sequence of length sn in which each of the integers 1, 2, 3, ..., n occurs exactly s times and between any two occurences of the integer i there are i entries. Thus, a perfect sequence for n is a generalized perfect 2-sequence.

The authors are unable to discover an n for which there is a generalized perfect s-sequence for $s > 2$ and pose as a problem the determination of all s and n for which there are generalized perfect s-sequences for n.

The following partial result is given in case $s = 3$. The method of proof becomes tedious for large n but could be settled for any given n on a machine.

Theorem. <u>There is no generalized perfect 3-sequence for</u> $n = 2, 3, 4, 5, 6$.

Proof. The case $n = 2$ is trivial.

Consider the case $n = 3$. Assume that there is a generalized perfect 3-sequence for $n = 3$. Beginning with the first 3 in the sequence we must have $3, a_1, a_2, a_3, 3$. There are 9 elements in the sequence including another 3 hence the entire sequence must be of the form

$$3, a_1, a_2, a_3, 3, b_1, b_2, b_3, 3 \quad .$$

The first occurence of 2 in the sequence is at a_2 or a_3 hence either $a_2 = b_1 = 2$ or $a_3 = b_2 = 2$ but in neither case is there room for another 2 and so there is no generalized perfect 3-sequence for $n = 3$.

Now, let $n = 4$. If the desired sequence is possible, beginning with the first 4 in the sequence we have $4, a_1, a_2, a_3, a_4, 4, b_1, b_2, b_3, b_4, 4$. Because of the positions of the 4's, a_1, a_2, a_4, b_1, b_4 are not 3 hence $a_3 = b_2 = 3$ and the sequence either begins or ends with a 3. Consider the case

$$4, a_1, a_2, 3, a_4, 4, b_1, 3, b_3, b_4, 4, 3 \quad .$$

the alternate case is similar. Because of the spaces already occupied by the 3's and 4's it is not possible to put the 2's in the sequence and so $n = 4$ is impossible.

In case $n = 5$, we must have the subsequence

$$5, a_1, a_2, a_3, a_4, a_5, 5, b_1, b_2, b_3, b_4, b_5, 5$$

in the proposed sequence. It is obvious that a_1, a_2, a_5 cannot be a 4. If $a_3 = 4$, the sequence is

(1) $$4, c_1, 5, a_1, a_2, 4, a_4, a_5, 5, b_1, 4, b_3, b_4, b_5, 5$$

or has

(2) $$5, a_1, a_2, 4, a_4, a_5, 5, b_1, 4, b_3, b_4, b_5, 5, 4$$

as a subsequence. If $a_4 = 4$, there is a subsequence

(3) $$4, 5, a_1, a_2, a_3, 4, a_5, 5, b_1, b_2, 4, b_4, b_5, 5$$

or the entire sequence is

(4) $$5, a_1, a_2, a_3, 4, a_5, 5, b_1, b_2, 4, b_4, b_5, 5, d_1, 4$$

For sequence (1), it is clear that one must have $a_1 = a_5 = b_3 = 3$ hence $a_4 = b_1 = b_4 = 2$ but this is impossible since one cannot have $c_1 = a_2 = b_5 = 1$. The argument for sequence (2) is the same.

For the sequence (3), the only possible choices for 3 make $a_3 = b_1 = b_5 = 3$. This done, the only choices for 2 make $a_2 = a_5 = b_2 = 2$ but this requires that $a_1 = b_4 = 1$ which is impossible. The argument for the sequence (4) is the same and it is seen that the case $n = 5$ cannot occur.

The case $n = 6$ is treated similarly. The details are numerous and will be omitted.

The authors are indebted to the referee for the following theorem.

Theorem. There is no generalized perfect s-sequence for $n < s$.

Proof. There are s terms equal to n, and between each of the $s-1$ pairs of adjacent n's is an interval of length n. The total length, $s + n(s-1)$, must not be greater than sn, which implies $n \geq s$.

REFERENCES

1. Davies, Roy O.: On Langford's problem (II), Math. Gaz. 43 (1959), pp. 253-255.

2. Langford, C. D.: Problem, Math. Gaz. 42 (1958), p. 228.

3. Priday, C. J.: On Langford's problem (I), Math. Gaz. 43 (1959), pp. 250-253.

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