DETERMINANTS INVOLVING Kth POWERS FROM SECOND ORDER SEQUENCES

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INTRODUCTION

Let a_n be a sequence of complex numbers satisfying the difference equation

(1) $$a_{n+2} = \alpha\, a_{n+1} - \beta\, a_n \quad \text{for} \quad n = 0, 1, \ldots ,$$

where α and β are fixed complex numbers, for such a sequence we define

(2) $$A_k(a_n) = \begin{vmatrix} a_n^k & a_{n+1}^k & \cdots & a_{n+k}^k \\ a_{n+1}^k & a_{n+2}^k & \cdots & a_{n+k+1}^k \\ \vdots & \vdots & & \vdots \\ a_{n+k}^k & a_{n+k+1}^k & \cdots & a_{n+2k}^k \end{vmatrix} \quad \text{for} \quad n = 0, 1, \ldots .$$

It is the purpose of this note to prove

(3) $$A_k(a_n) = \beta^{nk(k+1)/2} A_k(a_0) ,$$

and give examples of the result.

A DIFFERENCE EQUATION FOR $[a_n^k]$:

Let $(1 - \theta_1 x)(1 - \theta_2 x) = 1 - \alpha x + \beta x^2$, so that $\theta_1 + \theta_2 = \alpha$ and $\theta_1 \theta_2 = \beta$, and assume $\theta_1 \neq \theta_2$. Carlitz [3] has proved that

(4) $$p_k(x) / q_k(x) = \sum_{n=0}^{\infty} a_n^k x^n$$

where

(5) $$q_k(x) = \prod_{i=0}^{k} (1 - \theta_1^i \theta_2^{k-i} x)$$

and $p_k(x)$ is a polynomial of degree less than the degree of $q_k(x)$. Letting

(6) $$q_k(x) = 1 - \sum_{i=1}^{k} \alpha_{k+1-i}(k)\, x^i$$

(the constants $\alpha_j(k)$ are polynomials symmetric in θ_1 and θ_2 determined by (5)) we see after multiplying through (4) with $q_k(x)$ (as given in (6)) and equating coefficients of x^n in the right and left members that

(7) $$a_{n+k+1}^k = \alpha_{k+1}(k)\, a_{n+k}^k + \alpha_k(k)\, a_{n+k-1}^k + \cdots + \alpha_1(k) a_n^k$$

for $n = 0, 1, \ldots$. We also know from (5) and (6) that

(8) $$-\alpha_1(k) = (-1)^{k+1} (\theta_1 \theta_2)^{1+2+\cdots+k} \qquad \text{or}$$

$$\alpha_1(k) = (-1)^{k+2} \beta^{k(k+1)/2}$$

Now let k be a fixed natural number and consider for $n \geq 0$,

(9) $$(-1)^k (-1)^{k+2} \beta^{k(k+1)/2} A_k(a_n) = (-1)^k \alpha_1(k) A_k(a_n)$$

$$= \begin{vmatrix} a_{n+1}^k & a_{n+2}^k & \cdots & a_{n+k}^k & \alpha_1(k)\, a_n^k \\ a_{n+2}^k & a_{n+3}^k & \cdots & a_{n+k+1}^k & \alpha_1(k)\, a_{n+1}^k \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n+k+1}^k & a_{n+k+2}^k & \cdots & a_{n+2k}^k & \alpha_1(k)\, a_{n+k}^k \end{vmatrix} = A_k(a_{n+1}) \; .$$

The second equality in (9) follows since we have interchanged k columns and multiplied the last column by $\alpha_1(k)$. The last equality follows since appropriate multiples of the first columns can be added to the last column to make it the last column of $A_k(a_{n+1})$; the "appropriate multiples" are the $a_j(k)$ given in (7).

Thus, we have shown

$$(-1)^k(-1)^{k+2}\beta^{k(k+1)/2}A_k(a_n) = \beta^{k(k+1)/2}A_k(a_n) = A_k(a_{n+1}) \;,$$

so that (3) can be proved by induction on n.

As a corollary to (3) we note that if $\{a_n\}$ satisfies (1), then $\{a_{qn+p}\}$, where q and p are non-negative integers, is a second order sequence as well; in fact,

$$(10)\qquad a_{q(n+2)+p} = (\theta_1^q + \theta_2^q)\, a_{q(n+1)+p} - \beta^q a_{qn+p}$$

for $n = 0, 1, \ldots$ Hence we can rewrite (3) to obtain

$$(11)\qquad A_k(a_{qn+p}) = \beta^{qnk(k+1)/2} A_k(a_p) \;.$$

EXAMPLES INVOLVING THE FIBONACCI SEQUENCES

When a_n is the Fibonacci sequence $\{F_n\} = \{0, 1, 1, 2, \ldots\}$, $\beta = -1$ in (3) so that we have

$$(12)\qquad \begin{vmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{vmatrix} = (-1)^n \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^{n+1} \;,$$

$$(13)\qquad \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = (-1)^{3n} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 4 \\ 1 & 4 & 9 \end{vmatrix} = (-1)^{n+1}\, 2 \;,$$

$$(14)\qquad \begin{vmatrix} F_n^3 & F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 \\ F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 \\ F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 \\ F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 & F_{n+6}^3 \end{vmatrix} = (-1)^{6n} \begin{vmatrix} 0 & 1 & 1 & 8 \\ 1 & 1 & 8 & 27 \\ 1 & 8 & 27 & 125 \\ 8 & 27 & 125 & 512 \end{vmatrix} = 36$$

The result in (12) is well known, Brother Alfred proposed (13) as a problem in the very first issue of the Fibonacci Quarterly [1], and Erbacker, Fuchs and Parker proposed (14) in a later issue [5].

If we redefine $a_0 = F_1$, $a_1 = F_2$, ... we have $\{a_n\} = \{u_n\}$ in the standard notation; fixing $q = 2$ and $p = 1$ in (11) we obtain for $k = 1$ and 2,

(15) $$A_1(u_{2n+1}) = A_1(u_1) = -1 \quad ,$$

(16) $$A_2(u_{2n+1}) = A_2(u_1) = -18 \quad ,$$

respectively; on the other hand if we fix $q = 2$ and $p = 0$ in (11) we have for $k = 1$ and 2,

(17) $$A_1(u_{2n}) = A_1(u_0) = 1 \quad ,$$

(18) $$A_2(u_{2n}) = A_2(u_0) = 18 \quad ,$$

respectively. Together (16) and (18) imply

(19) $$\begin{vmatrix} u_n^2 & u_{n+2}^2 & u_{n+4}^2 \\ u_{n+2}^2 & u_{n+4}^2 & u_{n+6}^2 \\ u_{n+4}^2 & u_{n+6}^2 & u_{n+8}^2 \end{vmatrix} = (-1)^{n+1} \quad 18$$

which has also been proposed as a problem by Brother Alfred [2].

AN EXAMPLE INVOLVING A SEQUENCE OF POLYNOMIALS

Lorch and Moser [8] proposed that one prove

(20) $$\begin{vmatrix} v_n & v_{n+1} \\ v_{n+1} & v_{n+2} \end{vmatrix} = x \quad \text{for } n = 0, 1, 2, \ldots$$

where $v_0 = 1$ and

(21) $$v_n = \sum_{v=0}^{n} \binom{n+v}{n-v} x^v \quad \text{for } n = 1, 2, \ldots \quad .$$

In proving (2), Carlitz [4] proved

$$v_{n+2} = (x+2)\, v_{n+1} - v_n \quad \text{for } n = 0, 1, 2, \ldots \quad ; \tag{22}$$

hence, we can prove (20) and obtain generalizations by using (3). For $k = 1$ and 2 we have respectively,

$$A_1(v_n) = A_1(v_0) = x \quad , \tag{23}$$

$$A_2(v_n) = A_2(v_0) = 2x^3(x+2)^2 \quad . \tag{24}$$

A second generalization of this problem was also given by Gould [6].

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