RELATIVELY PRIME SEQUENCE SOLUTIONS OF NON-LINEAR DIFFERENCE EQUATIONS

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In the following we present a one-parameter family of first order non-linear difference equations which are shown to possess solutions which are sequences of (pair-wise) relatively prime numbers. In particular, we show that the sequences

(A) $F_k = 2^{2^k} + 1$

(B) $G_k^{\pm} = 2 \cosh \left[2^k \cosh^{-1} \frac{n_0}{2} \right] + \frac{1 \pm 3}{2}$,

where n_0 is an odd or even integer according as the sign in (B) is plus or minus, respectively, consist of relatively prime numbers by virtue of being solutions of such difference equations. Sequence (A) is the famous sequence of so-called Fermat numbers originally conjectured by Fermat to consist only of prime numbers (later disproved by Euler). The present method provides a new proof of the relatively primacy of these numbers (see [1], p. 14). Sequence (B) is apparently new and is the only other important solution which has been obtained in closed form.

These specific results are based on the following simple theorem: Theorem: If the sequence of integers $\{u_k\}$ satisfies the difference equation

(1) $$u_{k+1} = u_k^2 - bu_k + b$$

where $k = 0, 1, 2, \ldots$ and b is integral, then the distinct elements u_{k_1}, u_{k_2} have no common divisor except possibly for the divisors of b.

Proof. We may write equation (1) in the form

$$u_k - b = u_{k-1} (u_{k-1} - b) \quad ,$$

which by iteration may be expressed as (Continued on page 152.)