RELATIVELY PRIME SEQUENCE SOLUTIONS OF NON-LINEAR DIFFERENCE EQUATIONS

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In the following we present a one-parameter family of first
order non-linear difference equations which are shown to possess
solutions which are sequences of (pair-wise) relatively prime num-
bers. In particular, we show that the sequences

(A) \( F_k = 2^k + 1 \)

(B) \( G_k^\pm = 2 \cosh \left[ \frac{2^k \cosh^{-1} \frac{n_0}{2}}{2} \right] + \frac{1 \pm 3}{2} \),

where \( n_0 \) is an odd or even integer according as the sign in (B) is
plus or minus, respectively, consist of relatively prime numbers by
virtue of being solutions of such difference equations. Sequence (A)
is the famous sequence of so-called Fermat numbers originally con-
jectured by Fermat to consist only of prime numbers (later disproved
by Euler). The present method provides a new proof of the relatively
primacy of these numbers (see [1], p. 14). Sequence (B) is appar-
etly new and is the only other important solution which has been ob-
tained in closed form.

These specific results are based on the following simple theorem:

Theorem: If the sequence of integers \( \{ u_k \} \) satisfies the difference
equation

\[
(1) \quad u_{k+1} = u_k^2 - bu_k + b
\]

where \( k = 0, 1, 2, \ldots \) and \( b \) is integral, then the distinct elements
\( u_{k_1}, u_{k_2} \) have no common divisor except possibly for the divisors of

\( b, 1 \quad 1 \)

Proof. We may write equation (1) in the form

\[
u_k - b = u_{k-1} \left( u_{k-1} - b \right) ,
\]

which by iteration may be expressed as (Continued on page 152.)