ON EVALUATING CERTAIN COEFFICIENTS
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The coefficients to be discussed are those involved when expressing the general term of certain sequences, defined by difference equations, in terms of the roots of the related characteristic equation.

Case I: If the characteristic equation

\[ a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_{m-1} x + a_m = 0, \quad a_0 = 1 \]

has no multiple root then

\[ u_n = \sum_{k=1}^{m} C_k n^{n+1}, \quad u_n = 0, 1, 2, \ldots, \]

where \( x_k \), \( k = 1, 2, \ldots, m \), is a root of (1). If the boundary conditions are given by \( u_0 = u_1 = \ldots = u_{m-1} = 1 \) then

\[ C_k = \frac{\begin{vmatrix} x_1 x_2 \ldots x_{k-1} & x_k & x_{k+1} \ldots & x_m \\ x_1 x_2 \ldots x_{k-1} & 2 & 2 \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_1 x_2 \ldots x_{k-1} & m & m \ldots & m \end{vmatrix}}{\begin{vmatrix} x_1 x_2 \ldots x_{k-1} & x_k & x_{k+1} \ldots & x_m \\ x_1 x_2 \ldots x_{k-1} & 2 & 2 \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_1 x_2 \ldots x_{k-1} & m & m \ldots & m \end{vmatrix}} = \frac{N}{D} \]

Expanding the determinants and dividing common factors from the numerator and denominator gives
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(3) \[ N = (-1)^{k-1} \prod_{i=1, i \neq k}^{m} (x_i - 1) \]

(4) \[ D = (-1)^{k-1} \prod_{i=1}^{m} (x_i - x_k) \]

Since

\[ f(x) = \sum_{i=0}^{m} a_i x^{m-i} = \prod_{i=1}^{m} (x - x_i), \quad a_0 = 1, \quad a_1 = 1 \]

\[ f(1) = \prod_{i=1}^{m} (1 - x_i) \quad \text{and} \quad f'(x_k) = \prod_{i=1, i \neq k}^{m} (x_k - x_i) \]

Using these identities, (3) becomes

\[ N = \frac{(-1)^{m+k} f(1)}{(1 - x_k)}, \quad \text{if} \quad x_k \neq 1 \]

and (4) can be written

\[ D = (-1)^{m+k} x_k f'(x_k) \]

Substituting these in (2) gives

\[ C_k = 1, \quad x_k = 1 \]

\[ C_k = \frac{f(1)}{x_k (1 - x_k)f'(x_k)}, \quad x_k \neq 1 \]

Parker [4] investigated the general term of a recursive sequence and gives a method for determining these coefficients but does not give the general formula.

For the Fibonacci sequence the characteristic equation is

\[ x^2 - x - 1 = 0 \quad \text{and} \quad u_0 = u_1 = 1 \]
Therefore

\[ C_k = \frac{(-1)(-1)}{x_k(x_k - 1)(2x_k - 1)} = \frac{1}{2x_k - 1} \]

Some characteristic equations obtained in generalizations of the Fibonacci sequence and the values of \( C_k \) for each follow.

The characteristic equation in the generalization by Dickinson [1] is \( x^c - x^a - 1 = 0 \), \( a, c \) integers. Since

\[ x_k f'(x_k) = c x_k^c - a x_k^a = c(x_k^a + 1) - a x_k^a = (c - a)x_k^a + c \]

\[ C_k = \frac{1}{(x_k - 1) [(c - a)x_k^a + c]} \]

for the sequence in which \( u_0 = u_1 = \ldots = u_{c-1} = 1 \).

In the generalization by Harris and Styles [2] the characteristic equation is \( x^p(x - 1)^q - 1 = 0 \), \( p, q \) integers, \( p \geq 0 \), \( q \geq 1 \) and \( u_0 = u_1 = \ldots = u_{p+q-1} = 1 \).

\[ C_k = \frac{1}{(p + q)x_k^{p-q}} \]

as was shown in [2] without this formula.

Miles [3] used the characteristic equation

\[ x^k - x^{k-1} - \ldots - x - 1 = 0, \quad k \text{ integral } \geq 2 \]

For the sequence in which the initial conditions are given by

\[ u_0 = u_1 = \ldots = u_{k-1} = 1 \]

\[ C_j = \frac{k - 1}{2x_j^{k} - (k + 1)} \]

Raab [5] used the characteristic equation

\[ x^{r+1} - a x^r - b = 0, \quad a, b \text{ real, } \quad r \text{ integral } \geq 1 \]

For the sequence in which the initial conditions are given by

\[ u_0 = u_1 = \ldots = u_r = 1 \]
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\[
C_k = \frac{b + a - 1}{(a - 1) x_k^{r+1} + b [(r + 1) x_k^r - r]}
\]

The boundary conditions can be generalized slightly. If

\[
u_0 = pr, \; u_1 = pr^2, \; u_2 = pr^3, \; \ldots, \; u_n = pr^{n+1},
\]

\[
C_k = \frac{p f(r)}{x_k^{r+1} (1 - x_k) H(x_k^r)}
\]

Case II: If the characteristic equation (1) has a root of multiplicity 2 then

\[
u_n = (C_1 + 2C_2) x_2^{n+1} + \sum_{k=3}^{m} C_k x_k^{n+1}, \; n = 0, 1, 2, \ldots
\]

and \(x_2\) is the repeated root of (1). If the boundary conditions are given by \(u_0 = u_1 = \ldots = u_{m-1} = 1\), then,

\[
C_1 = \frac{N_1}{D}
\]

\[
\begin{bmatrix}
1 & x_2 & x_3 & \ldots & x_m \\
1 & 2x_2 & x_3 & \ldots & x_m \\
1 & 3x_2 & x_3 & \ldots & x_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & mx_2 & x_3 & \ldots & x_m \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_2 & x_2 & x_3 & \ldots & x_m \\
x_2 & 2x_2 & x_3 & \ldots & x_m \\
x_2 & 3x_2 & x_3 & \ldots & x_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_2 & mx_2 & x_3 & \ldots & x_m \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 2 & \ldots & \ldots & 2 \\
3 & 3 & \ldots & \ldots & 3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m & m & \ldots & \ldots & m \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}
\]

\[
D = \det \begin{bmatrix}
1 & x_2 & x_3 & \ldots & x_m \\
1 & 2x_2 & x_3 & \ldots & x_m \\
1 & 3x_2 & x_3 & \ldots & x_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & mx_2 & x_3 & \ldots & x_m \\
\end{bmatrix}
\]

\[
N_1 = \det \begin{bmatrix}
x_2 & x_2 & x_3 & \ldots & x_m \\
x_2 & 2x_2 & x_3 & \ldots & x_m \\
x_2 & 3x_2 & x_3 & \ldots & x_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_2 & mx_2 & x_3 & \ldots & x_m \\
\end{bmatrix}
\]
Expanding the determinants gives

\[
(6) \quad N_1 = \prod_{i=2}^{m-1} x_i \prod_{i=3}^{m-2} (x_{i-1} - x_i) \prod_{i=3}^{m} (x_{i-1} - x_i) \prod_{i=3}^{m} (x_{i-1} - x_i)
\]

\[
= \frac{(m-2)(m-3)}{2} \left( (x_4 - x_3) \cdot \left( \frac{m}{2} \right)^{-1} \right) \left( (2x_2 - 1) \cdot \prod_{i=3}^{m} (x_i - x_{i-1}) - x_2(x_2 - 1) \right)
\]

(the sum of all possible factors \(x_i - x_{i-1}\), \(i = 3, 4, \ldots, m\), taken \(m-3\) at a time)

Since

\[
f''(x_2) = 2 \prod_{i=3}^{m} (x_i - x_{i-1}) \quad \text{and} \quad f'''(x_2) = 6
\]

(the sum of all possible products of the factors \(x_i - x_{i-1}\), \(i = 3, 4, \ldots, m\), taken \(m-3\) at a time), the quantity in the braces in (6) can be expressed

\[
\frac{(m-1)(m-2)}{2} \cdot \left[ \frac{(2x_2 - 1)f''(x_2)}{2} + x_2(x_2 - 1)f'''(x_2) \right]
\]

Therefore,

\[
N_1 = (-1)^{m-2} \prod_{i=2}^{m} x_i \prod_{i=3}^{m-1} (x_{i-1} - x_i) \prod_{i=3}^{m} (x_{i-1} - x_i)
\]

\[
\prod_{i=3}^{m} (x_{i-1} - x_i) \prod_{i=3}^{m} (x_{i-1} - x_i) \left[ \frac{(2x_2 - 1)f''(x_2)}{2} + x_2(x_2 - 1)f'''(x_2) \right]
\]

Expanding the determinant in the denominator gives
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(8) \[ D = \prod_{i=2}^{m} x_1 \prod_{i=3}^{m} (x_i - x_2)^2 \prod_{i=4}^{m} (x_i - x_3) \prod_{i=5}^{m} (x_i - x_4) \cdots \]

\[ \prod_{i=m-1}^{m} (x_i - x_{m-2}) \prod_{i=m}^{m} (x_i - x_{m-1}) \]

Substituting (7) and (8) in (5) and simplifying gives

\[ C_1 = \frac{\prod_{i=3}^{m} (1 - x_i)}{x_2^{2} \prod_{i=3}^{m} (x_i - x_1)^2} \left[ \frac{2x_2 - 1}{2} f'(x_2) + \frac{x_2(x_2 - 1)}{6} f'''(x_2) \right] \]

If \( x_2 = 1 \), \( C_1 = 1 \). If \( x_2 \neq 1 \),

\[ C_1 = \frac{4(1-x_2)^2 \prod_{i=3}^{m} (1-x_i)}{x_2^{2} \left[ f'(x_2) \right]^2 (1-x_2)^2} \left[ \frac{(2x_2 - 1) f''(x_2)}{2} + \frac{x_2(x_2 - 1)}{6} f'''(x_2) \right] \]

Therefore,

\[ C_1 = \frac{2f(1)}{x_2(x_2 - 1) f''(x_2)} \left[ \frac{1}{x_2} + \frac{1}{x_2 - 1} + \frac{f'''(x_2)}{3 f''(x_2)} \right], \quad x_2 \neq 1 \]

To determine \( C_2 \), the numerator in (5) is replaced by

\[
\begin{array}{cccc}
  x_2 & x_3 & \cdots & x_m \\
  x_2 & x_3^2 & \cdots & x_m^2 \\
  x_2^2 & x_3^3 & \cdots & x_m^3 \\
  \vdots & \vdots & \ddots & \vdots \\
  x_2^m & x_3^m & \cdots & x_m^m \\
\end{array}
= N_2
\]
Evaluating gives

\[ N_2 = (-1)^m \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (1-x_i) \prod_{i=4}^{m} (x_i-x_2) \ldots \]

Dividing (9) by (8) and simplifying gives

\[ C_2 = \frac{(-1)^m \prod_{i=2}^{m} (1-x_i)}{m \prod_{i=3}^{m} (1-x_i)} = \frac{2 \prod_{i=3}^{m} (1-x_i)}{x_2^2 \prod_{i=3}^{m} (x_1-x_2)} \]

For \( x_2 = 1 \), \( C_2 = 0 \). For \( x_2 \neq 1 \),

\[ C_2 = \frac{2f(1)}{x_2^2 f'(x_2)(1-x_2)} \]

To determine \( C_k \), \( k = 3, 4, \ldots, m \), the numerator in (5) is replaced by

\[
\begin{array}{cccccccccccc}
  x_2 & x_2 & x_3 & \ldots & x_{k-1} & 1 & x_{k+1} & \ldots & x_m \\
  x_2 & 2x_2 & x_3 & \ldots & x_{k-1} & 1 & x_{k+1} & \ldots & x_2 \\
  x_2 & 3x_2 & x_3 & \ldots & x_{k-1} & 1 & x_{k+1} & \ldots & x_3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_2 & mx_2 & x_3 & \ldots & x_{k-1} & 1 & x_{k+1} & \ldots & x_m \\
\end{array} = N_k
\]

Evaluating \( N_k \) yields
Substituting (11) and (8) in (10) leads to

\[ C_k = \frac{(-1)^k m}{k-1} \frac{m}{i=3} \frac{m}{i \neq k} \frac{m}{i=2} \frac{m}{i=k+1} \frac{m}{i=2} \frac{m}{i=k+1} \]

\[ x_k (1-x_k) \frac{m}{i=2} \frac{m}{i=k+1} \frac{m}{x_k (1-x_k) (1-x_k)^{-1} m-k} \frac{m}{(x_k-x_i)} \frac{m}{(x_k-x_i)} \]

There \( C_k = 1, x_k = 1. \)

\[ C_k = \frac{f(1)}{x_k (1-x_k)^{m-k} f'(x_k)} \text{ if } x_k \neq 1. \]

Summary: If the roots of

\[ f(x) = \sum_{i=0}^{m} a_i x^{m-1} = 0 \]

are not repeated and \( u_0 = u_1 = u_2 = \ldots = u_{m-1} = 1, \)
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\[ C_k = \begin{cases} 
1, & x_k = 1 \\
\frac{f(1)}{x(1-x_k)f'(x_k)}, & x_k \neq 1
\end{cases} \]

If \( f(x) \) has a double root, \( x_1 = x_2 \), and \( u_0 = u_1 = \ldots = u_{m-1} = 1 \),

\[
C_1 = \begin{cases} 
1, & x_2 = 1 \\
\frac{2f(1)}{x_2(x_2-1)f'(x_2)} \left[ \frac{1}{x_2} + \frac{1}{x_2-1} + \frac{f'''(x_2)}{3f''(x_2)} \right], & x_2 \neq 1
\end{cases}
\]

\[
C_2 = \begin{cases} 
0, & x_2 = 1 \\
\frac{2f(1)}{x_2(x_2-1)f'(x_2)}, & x_2 \neq 1
\end{cases}
\]

\[
C_k = \begin{cases} 
1, & x_k = 1 \\
\frac{f(1)}{x_k f'(x_k)(1-x_k)}, & x_k \neq 1
\end{cases} \quad k = 3, 4, \ldots, m
\]

REFERENCES


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