ON EVALUATING CERTAIN COEFFICIENTS

Carolyn C. Styles
San Diego Mesa College, San Diego, California

The coefficients to be discussed are those involved when expressing the general term of certain sequences, defined by difference equations, in terms of the roots of the related characteristic equation.

Case I: If the characteristic equation

$$(1)\qquad a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \dots + a_{m-1}x + a_m = 0,\quad a_0 = 1\ ,$$

has no multiple root then

$$u_n = \sum_{k=1}^{m} C_k x_k^{n+1}\ ,\quad u_n = 0, 1, 2, \dots,$$

where x_k, $k = 1, 2, \dots, m$, is a root of (1). If the boundary conditions are given by $u_0 = u_1 = \dots = u_{m-1} = 1$ then

$$(2)\qquad C_k = \frac{\begin{vmatrix} x_1 x_2 & \dots & x_{k-1} & 1 & x_{k+1} \dots & x_m \\ x_1^2 x_2^2 & \dots & x_{k-1}^2 & 1 & x_{k+1}^2 \dots & x_m^2 \\ \cdot\ \cdot & & \cdot & & \cdot\ \cdot & \cdot \\ \cdot\ \cdot & & \cdot & & \cdot\ \cdot & \cdot \\ \cdot\ \cdot & & \cdot & & \cdot\ \cdot & \cdot \\ x_1^m x_2^m & & x_{k-1}^m & 1 & x_{k+1}^m \dots & x_m^m \end{vmatrix}}{\begin{vmatrix} x_1 x_2 & \dots & x_{k-1} & x_k & x_{k+1} & \dots & x_m \\ x_1^2 x_2^2 & \dots & x_{k-1}^2 & x_k^2 & x_{k+1}^2 & \dots & x_m^2 \\ \cdot\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ x_1^m x_2^m \dots & & x_{k-1}^m & x_k^m & x_{k+1}^m & \dots & x_m^m \end{vmatrix}} = \frac{N}{D}$$

Expanding the determinants and dividing common factors from the numerator and denominator gives

(3) $$N = (-1)^{k-1} \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_i - 1)$$

(4) $$D = (-1)^{k-1} x_k \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_i - x_k)$$

Since

$$f(x) = \sum_{i=0}^{m} a_i x^{m-i} = \prod_{i=1}^{m} (x - x_i), \quad a_0 = 1 \quad ,$$

$$f(1) = \prod_{i=1}^{m} (1 - x_i) \quad \text{and} \quad f'(x_k) = \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_k - x_i) \quad ,$$

Using these identities, (3) becomes

$$N = \frac{(-1)^{m+k} f(1)}{(1 - x_k)} , \quad \text{if } x_k \neq 1$$

and (4) can be written

$$D = (-1)^{m+k} x_k f'(x_k)$$

Substituting these in (2) gives

$$C_k = 1, \quad x_k = 1$$

$$C_k = \frac{f(1)}{x_k(1 - x_k)f'(x_k)} , \quad x_k \neq 1 \quad .$$

Parker [4] investigated the general term of a recursive sequence and gives a method for determining these coefficients but does not give the general formula.

For the Fibonacci sequence the characteristic equation is

$$x^2 - x - 1 = 0 \quad \text{and} \quad u_0 = u_1 = 1 \quad .$$

Therefore

$$C_k = \frac{(-1)(-1)}{x_k(x_k - 1)(2x_k - 1)} = \frac{1}{2x_k - 1}$$

Some characteristic equations obtained in generalizations of the Fibonacci sequence and the values of C_k for each follow.

The characteristic equation in the generalization by Dickinson [1] is $x^c - x^a - 1 = 0$, a, c integers. Since

$$x_k f'(x_k) = cx_k^c - ax_k^a = c(x_k^a + 1) - ax_k^a = (c - a)x_k^a + c \quad ,$$

$$C_k = \frac{1}{(x_k - 1)\left[(c - a)x_k^a + c\right]}$$

for the sequence in which $u_0 = u_1 = \ldots = u_{c-1} = 1$.

In the generalization by Harris and Styles [2] the characteristic equation is $x^p(x - 1)^q - 1 = 0$, p, q integers, $p \geq 0$, $q \geq 1$ and $u_0 = u_1 = \ldots = u_{p+q-1} = 1$.

$$C_k = \frac{1}{(p + q)x_{k-p}}$$

as was shown in [2] without this formula.

Miles [3] used the characteristic equation

$$x^k - x^{k-1} - \ldots - x - 1 = 0, \quad k \text{ integral} \geq 2 \quad .$$

For the sequence in which the initial conditions are given by

$$u_0 = u_1 = \ldots = u_{k-1} = 1 \quad ,$$

$$C_j = \frac{k - 1}{2x_j^k - (k + 1)} \quad , \quad j = 1, 2, \ldots, k \quad .$$

Raab [5] used the characteristic equation

$$x^{r+1} - ax^r - b = 0, \ a, b \text{ real}, \quad r \text{ integral} \geq 1 \quad .$$

For the sequence in whichthe initial conditions are given by

$$u_0 = u_1 = \ldots = u_r = 1 \quad ,$$

$$C_k = \frac{b + a - 1}{(a - 1)x_k^{r+1} + b\left[(r + 1)x_k - r\right]}$$

The boundary conditions can be generalized slightly. If

$$u_0 = pr,\ u_1 = pr^2,\ u_2 = pr^3,\ \ldots,\ u_n = pr^{n+1}\ ,$$

$$C_k = \frac{pf(r)}{x_k(1 - x_k)f'(x_k)}$$

Case II: If the characteristic equation (1) has a root of multiplicity 2 then

$$u_n = (C_1 + 2C_2)x_2^{n+1} + \sum_{k=3}^{m} C_k x_k^{n+1}\ ,\quad n = 0, 1, 2, \ldots$$

and x_2 is the repeated root of (1). If the boundary conditions are given by $u_0 = u_1 = \ldots = u_{m-1} = 1$, then,

$$(5)\qquad C_1 = \frac{\begin{vmatrix} 1 & x_2 & x_3 & \cdots & x_m \\ 1 & 2x_2^2 & x_3^2 & \cdots & x_m^2 \\ 1 & 3x_2^3 & x_3^3 & \cdots & x_m^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & mx_2^m & x_3^m & \cdots & x_m^m \end{vmatrix}}{\begin{vmatrix} x_2 & x_2 & x_3 & \cdots & x_m \\ x_2^2 & 2x_2^2 & x_3^2 & \cdots & x_m^2 \\ x_2^3 & 3x_2^3 & x_3^3 & \cdots & x_m^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_2^m & mx_2^m & x_3^m & \cdots & x_m^m \end{vmatrix}} = \frac{N_1}{D}$$

Expanding the determinants gives

$$(6)\quad N_1 = \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (x_i - 1) \prod_{i=3}^{m-1} (x_m - x_i) \prod_{i=3}^{m-2} (x_{m-1} - x_i) \ldots \prod_{i=3}^{4} (x_5 - x_i)$$

$$(x_4 - x_3) \Bigg\{ (-1)^{\frac{(m-2)(m-3)}{2}} \left[(2x_2 - 1) \prod_{i=3}^{m} (x_i - x_2) - x_2(x_2 - 1) \right.$$

(the sum of all possible factors $(x_i - x_2)$, $i = 3, 4, \ldots, m$, taken $m-3$ at a time$\left.\right]\Bigg\}$

Since

$$f''(x_2) = 2 \prod_{i=3}^{m} (x_2 - x_i) \quad \text{and} \quad f'''(x_2) = 6$$

(the sum of all possible products of the factors $(x_2 - x_i)$, $i = 3, 4, \ldots, m$, taken $m-3$ at a time), the quantity in the braces in (6) can be expressed

$$(-1)^{\frac{(m-1)(m-2)}{2}} \left[\frac{(2x_2 - 1)f''(x_2)}{2} + x_2(x_2 - 1)\frac{f'''(x_2)}{6} \right]$$

Therefore,

$$N_1 = (-1)^{\frac{(m-1)(m-2)}{2}} \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (x_i - 1) \prod_{i=3}^{m-1} (x_m - x_i)$$

$$\prod_{i=3}^{m-2} (x_{m-1} - x_i) \ldots \prod_{i=3}^{4} (x_5 - x_i) \cdot (x_4 - x_3) \left[\frac{(2x_2 - 1)f''(x_2)}{2} \right.$$

$$\left. + x_2(x_2 - 1) \frac{f'''(x_2)}{6} \right]$$

Expanding the determinant in the denominator gives

$$(8)\qquad D = x_2^2 \prod_{i=2}^{m} x_i \prod_{i=3}^{m} (x_i - x_2)^2 \prod_{i=4}^{m} (x_i - x_3) \prod_{i=5}^{m} (x_i - x_4) \cdots$$

$$\prod_{i=m-1}^{m} (x_i - x_{m-2}) \prod_{i=m}^{m} (x_i - x_{m-1})$$

Substituting (7) and (8) in (5) and simplifying gives

$$C_1 = \frac{\prod_{i=3}^{m} (1 - x_i)}{x_2^2 \prod_{i=3}^{m} (x_2 - x_i)^2} \left[\frac{2x_2 - 1)f''(x_2)}{2} + \frac{x_2(x_2-1)f'''(x_2)}{6}\right]$$

If $x_2 = 1$, $C_1 = 1$. If $x_2 \neq 1$,

$$C_1 = \frac{4(1-x_2)^2 \prod_{i=3}^{m} (1-x_i)}{x_2^2 \left[f''(x_2)\right]^2 (1-x_2)^2} \left[\frac{(2x_2-1)f''(x_2)}{2} + \frac{x_2(x_2-1)f'''(x_2)}{6}\right]$$

Therefore,

$$C_1 = \frac{2f(1)}{x_2(x_2-1)f''(x_2)} \left[\frac{1}{x_2} + \frac{1}{x_2 - 1} + \frac{f'''(x_2)}{3f''(x_2)}\right], \quad x_2 \neq 1 .$$

To determine C_2 the numerator in (5) is replaced by

$$\begin{vmatrix} x_2 & 1 & x_3 & \cdots & x_m \\ x_2^2 & 1 & x_3^2 & \cdots & x_m^2 \\ x_2^3 & 1 & x_3^3 & & x_m^3 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ x_2^m & 1 & x_3^m & \cdots & x_m^m \end{vmatrix} = N_2$$

Evaluating gives

$$(9) \qquad N_2 = (-1)^m \prod_{i=2}^{m} x_i \prod_{i=2}^{m} (1 - x_i) \prod_{i=3}^{m} (x_i - x_2) \prod_{i=4}^{m} (x_1 - x_3) \cdots \prod_{i=m-1}^{m} (x_i - x_{m-2}) \prod_{i=m}^{m} (x_i - x_{m-1})$$

Dividing (9) by (8) and simplifying gives

$$C_2 = \frac{(-1)^m \prod_{i=2}^{m} (1 - x_i)}{x_2^2 \prod_{i=3}^{m} (x_i - x_2)} = \frac{2 \prod_{i=2}^{m} (1 - x_i)}{x_2^2 f''(x_2)}$$

For $x_2 = 1$, $C_2 = 0$. For $x_2 \neq 1$,

$$C_2 = \frac{2f(1)}{x_2^2 f''(x_2)(1 - x_2)} \quad .$$

To determine C_k, $k = 3, 4, \ldots, m$, the numerator in (5) is replaced by

$$\begin{vmatrix} x_2 & x_2 & x_3 & \cdots & x_{k-1} & 1 & x_{k+1} & \cdots & x_m \\ x_2^2 & 2x_2^2 & x_3^2 & \cdots & x_{k-1}^2 & 1 & x_{k+1}^2 & \cdots & x_m^2 \\ x_2^3 & 3x_2^3 & x_3^3 & \cdots & x_{k-1}^3 & 1 & x_{k+1}^3 & \cdots & x_m^3 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ x_2^m & mx_2^m & x_3^m & \cdots & x_{k-1}^m & 1 & x_{k+1}^m & \cdots & x_m^m \end{vmatrix} = N_k$$

Evaluating N_k yields

$$(11)\quad N_k = (-1)^{k-1} x_2^3 \prod_{\substack{i=3\\ i\neq k}}^{m} x_i \prod_{\substack{i=2\\ i\neq k}}^{m} (1-x_i) \prod_{\substack{i=2\\ i\neq k}}^{m-1} (x_m - x_i) \prod_{\substack{i=2\\ i=k}}^{m-2} (x_{m-1} - x_i) \cdots$$

$$\prod_{\substack{i=2\\ i\neq k}}^{k+1} (x_{k+2} - x_i) \prod_{i=2}^{k-1} (x_{k+1} - x_i) \cdots \prod_{i=2}^{3} (x_4 - x_i) \prod_{i=2}^{2} (x_3 - x_i)$$

$$(1-x_2) \prod_{\substack{i=3\\ i\neq k}}^{m} (x_i - x_2)$$

Substituting (11) and (8) in (10) leads to

$$C_k = \frac{(-1)^{k+m}(1-x_2)^2 \prod_{\substack{i=3\\ i\neq k}}^{m} (1-x_i)}{x_k(x_k - x_2) \prod_{i=2}^{k-1} (x_k - x_i) \prod_{i=k+1}^{m} (x_i - x_k)}$$

$$C_k = \frac{(-1)^{k+m}(1-x_2)^2 \prod_{i=3}^{m} (1-x_i)}{x_k(1-x_k) \prod_{i=2}^{k-1} (x_k - x_i)(-1)^{m-k} \prod_{i=k+1}^{m} (x_k - x_i)}, \quad x_k \neq 1$$

There $C_k = 1$, $x_k = 1$.

$$C_k = \frac{f(1)}{x_k(1-x_k)f'(x_k)} \quad \text{if } x_k \neq 1 .$$

Summary: If the roots of

$$f(x) = \sum_{i=0}^{m} a_i x^{m-1} = 0$$

are not repeated and $u_0 = u_1 = u_2 = \cdots = u_{m-1} = 1$,

$$C_k = \begin{cases} 1 & , \quad x_k = 1 \\ \dfrac{f(1)}{x(1-x_k)f'(x_k)} & , \quad x_k \neq 1 \end{cases}$$

If $f(x)$ has a double root, $x_1 = x_2$, and $u_0 = u_1 = \cdots = u_{m-1} = 1$,

$$C_1 = \begin{cases} 1 & , \quad x_2 = 1 \\ \dfrac{2f(1)}{x_2(x_2-1)f''(x_2)} \left[\dfrac{1}{x_2} + \dfrac{1}{x_2-1} + \dfrac{f'''(x_2)}{3f''(x_2)} \right] & , \quad x_2 \neq 1 \end{cases}$$

$$C_2 = \begin{cases} 0 & , \quad x_2 = 1 \\ \dfrac{2f(1)}{x_2^2(1-x_2)f''(x_2)} & , \quad x_2 \neq 1 \end{cases}$$

$$C_k = \begin{cases} 1 & , \quad x_k = 1 \\ \dfrac{f(1)}{x_k f'(x_k)(1-x_k)} & , \quad x_k \neq 1 \end{cases} , \qquad k = 3, 4, \ldots, m$$

REFERENCES

1. Dickinson, David, "On Sums Involving Binomial Coefficients," American Mathematical Monthly, Vol. 57, 1950, pp. 82-86.
2. Harris, V. C. and Styles, C. C., "A Generalization of Fibonacci numbers," The Fibonacci Quarterly, Vol. 2, No. 4, Dec. 1964.
3. Miles, E. P., Jr., "Generalized Fibonacci Sequences," American Mathematical Monthly, Vol. 67, 1960, pp. 745-752.
4. Parker, Francis D., "On the General Term of a Recursive Sequence," The Fibonacci Quarterly, Vol. 2, No. 1, Feb. 1964, p. 67.
5. Raab, Joseph A., "A Generalization of the Connection Between the Fibonacci Sequences and Pascal's Triangle," The Fibonacci Quarterly, Vol. 1, 1963, pp. 21-31.

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