OPTIMALITY PROOF FOR THE SYMMETRIC FIBONACCI SEARCH TECHNIQUE

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An important problem in engineering, economics, and statistics is to find the maximum of a function. When the function has only one stationary point, the maximum, and when it depends on a single variable in a finite interval, the most efficient way to find the maximum is based on the Fibonacci numbers. The procedure, now known widely as "Fibonacci search," was discovered and shown optimal in a minimax sense by Kiefer [1]. S. Johnson [2] gave a different demonstration. Oliver and Wilde [3,4] extended the procedure to the case where, in order to distinguish between adjacent measurements, a non-negligible distance must be preserved between them. Although this modification, called the symmetric Fibonacci technique is described informally in [4], where numerous extensions are also discussed the present paper gives the first complete, precise description, and formal optimality proof.

Let $y(x)$ be a unimodal function, i.e., one with a unique relative maximum value obtained at $x = x^*$ on the closed interval $[a,b]$; thus

$$y(x^*) = \max_{a \le x \le b} y(x) \tag{1}$$

and $a \le x_1 < x_2 \le x^*$ implies $y(x_1) < y(x_2)$; $x^* \le x_1 < x_2 \le b$ implies $y(x_1) > y(x_2)$. As a practical consideration, assume that $y(x)$ can be measured only with finite accuracy, but there is a $\delta > 0$ such that if $|x_1 - x_2| \ge \delta$ the measurements of $y(x_1)$ and $y(x_2)$ will be equal only if x_1 and x_2 lie on opposite sides of x^*.

A search strategy $S(n, \delta)$ on $[a,b]$ is a plan for evaluating the function at n distinct points $x_1, x_2, \cdots, x_n$, where the location of x_{k+1} depends on $y(x_j)$ for $j \le k$ and where

$$|x_j - x_k| \ge \delta \qquad j \ne k, \qquad 1 \le j, \quad k \le n \tag{2}$$

The plan terminates after successive reduction of a starting interval on which the function is defined to a final interval of a required length, containing x^*.

Suppose k function evaluations have been performed, and let m_k be such that

(3) $$y(m_k) = \max\{y(x_1), \cdots, y(x_k)\}$$

Let

(4) $$L_k = \{x_j : x_j < m_k, \ j \le k\} \cup \{a\}$$

(5) $$R_k = \{x_j : x_j \geq m_k, \ j \le k\} \cup \{b\}$$

and

(6) $$\ell_k = \text{Sup } L_k$$

(7) $$r_k = \text{Inf } R_k$$

Since y is unimodal, it is clear that x^* must be in the interval $[\ell_k, r_k]$.

The purpose of this paper is to derive a search strategy $S_{op}(n, \delta)$ that is optimal in the sense that it maximizes the length of the starting interval for whatever final interval is given. From (2) - (7) it follows that the final interval for any $n > 1$ is

(8) $$r_n - \ell_n \geq 2\delta$$

By choosing the required length $(\geq 2\delta)$ of the final interval we rescale the system as follows:

(9) $$\bar{r}_n - \bar{\ell}_n = \lambda(r_n - \ell_n) = 1, \qquad \lambda > 0$$

(10) $$\bar{\delta} = \lambda\delta$$

(11) $$\bar{x} = \lambda(x - a)$$

Thus the new strategy $\bar{S}(n, \bar{\delta})$ is defined on $[0, \bar{D}_n]$, where $\bar{D}_n = \lambda(b - a)$, and it yields a final interval of unit length.

The <u>symmetric Fibonacci search</u> is a search strategy for a starting interval $[0, \bar{D}_n^F]$ where the right end point $\bar{D}_n^F$ is given by

$$\bar{D}_n^F = F_{n+1} - F_{n-1}\delta, \quad n = 1, 2, \cdots \tag{12}$$

where

$$F_{n+2} = F_{n+1} + F_n \quad (F_0 = 0, \ F_1 = 1) \tag{13}$$

are the Fibonacci numbers. The strategy is defined by the rules

$$\bar{x}_1^F = F_n - F_{n-2}\bar{\delta} \tag{14}$$

and

$$\bar{x}_{k+1}^F = \bar{\ell}_k + \bar{r}_k - \bar{m}_k \tag{15}$$

<u>Lemma:</u> The symmetric Fibonacci search defined above is an $\bar{S}(n, \bar{\delta})$ search strategy for $[0, \bar{D}_n^F]$ with $\bar{r}_n - \bar{\ell}_n = 1$.

<u>Proof:</u> By induction on n. The lemma is trivially true for $n = 1$, since by (3) - (7) $\bar{\ell}_1 = 0$ and $\bar{r}_1 = 1$. For $n = 2$, $\bar{D}_2^F = 2 - \bar{\delta}$ and $\bar{x}_1^F = 1$, implying that $\bar{\ell}_1 = 0$, $\bar{r}_1 = 2 - \bar{\delta}$ and consequently $\bar{x}_2^F = 1 - \bar{\delta}$ by (15). Thus $|\bar{x}_1^F - \bar{x}_2^F| = \bar{\delta}$ and for any unimodal function either $\bar{m}_2 = \bar{x}_1^F$ or $\bar{m}_2 = \bar{x}_2^F$, and in both cases $\bar{r}_2 - \bar{\ell}_2 = 1$.

Now assume the lemma true for $n = N$ (≥ 2). Then $\bar{D}_{N+1}^F = F_{N+2} - F_N\bar{\delta}$ and $\bar{x}_1^F = F_{N+1} - F_{N-1}\bar{\delta} = \bar{D}_N^F$. Consequently, $\bar{\ell}_1 = 0$, $\bar{r}_1 = F_{N+2} - F_N\bar{\delta}$ and $\bar{x}_2^F = F_N - F_{N-2}\bar{\delta} = \bar{D}_{N-1}^F$. Also $|\bar{x}_1^F - \bar{x}_2^F| = \bar{D}_N^F - \bar{D}_{N-1}^F \geq 1 - \bar{\delta} \geq \bar{\delta}$. If $\bar{m}_2 = \bar{x}_2^F$ then $\bar{\ell}_2 = 0$ and $\bar{r}_2 = \bar{x}_1^F = \bar{D}_N^F$. Thus $0 \leq \bar{x}^* \leq \bar{D}_N^F$ with one evaluation of the function at $\bar{D}_{N-1}^F$ and G_y the induction hypothesis the lemma is true for this case. If $\bar{m}_2 = \bar{x}_1^F$, then $\bar{\ell}_2 = \bar{x}_2^F = \bar{D}_{N-1}^F$ and $\bar{r}_2 = \bar{D}_{N+1}^F$. Define the new variable $\hat{x}^F = \bar{D}_{N+1}^F - \bar{x}^F$ so that $0 \leq \hat{x}^F \leq \bar{D}_N^F$ with one function evaluation at $\bar{D}_{N-1}^F$. Thus by induction the lemma is true for this case, too.

Note that if $y(\bar{x}_{k+1}^F) = y(\bar{m}_k)$ and say $\bar{m}_k > \bar{x}_{k+1}^F$, $\bar{x}^*$ is known to lie in the interval $[\bar{x}_{k+1}^F, \bar{m}_k]$. As $\bar{m}_{k+1} = \bar{m}_k$ or $\bar{m}_{k+1} = \bar{x}_{k+1}^F$, the Fibonacci search plan suggests reducing the interval $[\bar{\ell}_k, \bar{r}_k]$ to either $[\bar{\ell}_k, \bar{m}_k]$ or $[\bar{x}^F_{k+1}, \bar{r}_k]$. An improved choice could be made in this case by taking the next interval as $[\bar{\ell}_{k+1}, \bar{r}_{k+1}] = [\bar{x}^F_{k+1}, \bar{m}_k]$ which can be the starting interval for a new $\bar{S}(n - k - 2, \bar{\delta})$ Fibonacci plan. This fortuitous situation is deliberately excluded from the theorem following in order to simplify the proof.

Theorem: The symmetric Fibonacci search plan is the unique $\bar{S}_{op}(n, \bar{\delta})$ strategy among all $\bar{S}(n, \bar{\delta})$ strategies, provided $y(\bar{x}_{k+1}) \neq y(m_k)$ for every $k = 1, 2, \cdots, n - 1$.

Proof: by induction on n. For $n = 1$ the proof is obvious because all $\bar{D}_1 = \bar{D}^F_1 = 1$. For $n = 2$ we can assume without loss of generality that $\bar{x}_1 > \bar{x}_2$. Then any $\bar{S}(n, \bar{\delta})$ must satisfy $\bar{x}_1 = 1$, $\bar{D}_2 - \bar{x}_1 \leq 1 - \bar{\delta}$, yielding $\bar{D}_2 \leq 2 - \bar{\delta} = \bar{D}^F_2$. Assume now that the theorem is true for $n = N \geq 2$, then for an $\bar{S}(N + 1, \bar{\delta})$ strategy let $\bar{x}_1 > \bar{x}_2$ and $\bar{m}_2 = \bar{x}_2$. Hence $\bar{\ell}_2 = 0$, $\bar{r}_2 = \bar{x}_1$, and we have an $\bar{S}(N, \bar{\delta})$ strategy on $[0, \bar{x}_1]$. By induction

(16) $$\bar{x}_1 \leq \bar{D}^F_N$$

If $\bar{m}_2 = \bar{x}_1$, then $\bar{\ell}_2 = \bar{x}_2$, $\bar{r}_2 = \bar{D}_{N+1}$, and by induction therefore

(17) $$\bar{D}_{N+1} - \bar{x}_2 \leq \bar{D}^F_N$$

However,

(18) $$\bar{x}_2 \leq \bar{D}^F_{N-1}$$

also by induction, since if $\bar{x}^* < \bar{x}_2$ then $\bar{x}^*$ has to be located by an $\bar{S}(N - 1, \bar{\delta})$ strategy on $[0, \bar{x}_2]$. Thus addition of (17) and (18) yields

(19) $$\bar{D}_{N+1} \leq \bar{D}^F_N + \bar{D}^F_{N-1}$$

But $\bar{D}^F_N + \bar{D}^F_{N-1} = \bar{D}^F_{N+1}$ by (12), and this shows that the symmetric Fibonacci search is indeed optimal.

Moreover, we observe that

(20) $$\bar{D}_{N+1} - \bar{x}_1 \leq \bar{D}^F_{N-1}$$

by an argument similar to that used in (18). Relations (16) - (20) show that the symmetric Fibonacci search is the only way to achieve the maximum value $\overline{D}_N = \overline{D}_N^F$.

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ALGEBRA THROUGH PROBLEM SOLVING, Book Review, Cont'd from p. 264.

From the standpoint of the Fibonacci association, this text is a landmark in its recognition of the pedagogical value of the Fibonacci series. Let us hope that other authors will see the wisdom of incorporating such interesting and pregnant material into their textbooks.

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