ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. Hillman, University of New Mexico, Albuquerque, New Mex.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-94 ***Proposed by Clyde A. Bridger, Springfield Jr. College, Springfield, Ill***

Show that the number N_n of non-zero terms in the expansion of

$$K_n = \begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & & & & & & & & \\ \cdots & & & & & & & & \\ \cdots & & & & & & & & \\ 0 & 0 & 0 & & \cdots & 0 & -1 & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & & \cdots & 0 & 0 & -1 & a_n \end{vmatrix}$$

is obtained by replacing each a_i and each b_i by 1 and evaluating K_n. Show further that $N_n = F_{n+1}$, the $(n + 1)$st Fibonacci number.

B-95 ***Proposed by Brother U. Alfred, St. Mary's College, Calif.***

What is the highest power of 2 that exactly divides

$$F_1F_2F_3 \cdots F_{100} \quad ?$$

B-96 ***Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mex.***

Let G_n be the number of ways of expressing the positive integer n as an ordered sum $a_1 + a_2 + \cdots + a_s$ with each a_i in the set $\{1,2,3\}$. (For example, $G_3 = 4$ since 3 has just the expressions $3, 2+1, 1+2, 1+1+1.$) Find and prove the lowest order linear homogeneous recursion relation satisfied by the G_n.

B-97 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Let $A = \{a_k\}_{k=1}^{\infty}$ be an increasing sequence of numbers and let $A(n)$ denote the number of terms of A not greater than n. The Schnirelmann density of A is defined as the greatest lower bound of the ratios $A(n)/n$ for $n = 1, 2, \cdots$. Show that the Fibonacci sequence has density zero.

B-98 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Let F_n be the n^{th} Fibonacci number and find a compact expression for the sum $S_n(x) = F_1x + F_2x^2 + F_3x^3 + \cdots + F_nx^n$.

B-99 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Find a compact expression for the infinite sum

$$T(x) = S_1(x) + \frac{S_2(x)}{2!} + \frac{S_3(x)}{3!} + \cdots ,$$

where $S_n(x)$ is as defined in B-98.

SOLUTIONS

DIFFERENCE AND DIFFERENTIAL EQUATIONS

B-76 ***Proposed by James A. Jeske, San Jose State College, San Jose, Calif.***

The recurrence relation for the sequence of Lucas numbers is $L_{n+2} - L_{n+1} - L_n = 0$ with $L_1 = 1$, $L_2 = 3$.

Find the transformed equation, the exponential generating function, and the general solution.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

By (4.2) of Jeske's article, the transformed equation is

$$L_2(D)Y = Y'' - Y' - Y = 0, \quad Y(0) = 2, \quad Y'(0) = 1 \quad .$$

Now $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$ are the roots of $L_2(r) = 0$, hence with the given initial values we may determine the solution for the Lucas sequence to be

$$L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n \quad .$$

The exponential generating function for the Lucas sequence is thus

$$Y(t) = e^{r_1 t} + e^{r_2 t} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!} \quad .$$

Also solved by Clyde A. Bridger, Howard L. Walton, and the Proposer.

IT PAYS TO CHECK

B-77 ***Proposed by James A. Jeske, San Jose State College, San Jose, Calif.***

Find the general solution and the exponential generating function for the recurrence relation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0 \ ,$$

with $y_0 = 0$, $y_1 = 0$, and $y_2 = -1$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Since $L_3(x) = x^3 - 5x^2 + 8x - 4 = (x - 1)(x - 2)(x - 2) = 0$, we have, using the notation of section 4 of Jeske's paper, $r_1 = 1$, $m_1 = 1$, $r_2 = 2$,

$m_2 = 2$, and $m = 2$. With the given initial values we find the exponential generating function to be

$$Y(t) = -e^t + e^{2t}(1 - t)$$

which, by applying the inverse transform (3.3), yields the general solution as

$$y_n = 2^n\left(1 - \frac{n}{2}\right) - 1 .$$

Also solved by Clyde A. Bridger and the Proposer. Douglas Lind also noted that formula (4.8) of Jeske's paper is not correct.

A LUCAS SUM

B-78 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Show that

$$F_n = L_{n-2} + L_{n-6} + \cdots + L_{n-2-4m} + e_n, \quad n > 2 ,$$

where m is the greatest integer in $(n - 3)/4$, and $e_n = 0$ if $n \equiv 0 \pmod 4$, $e_n = 1$ if $n \not\equiv 0 \pmod 4$.

Solution by the Proposer.

Proof by induction: The proposition is easily shown true for $n = 3, 4, 5, 6$. Now assume the theorem true for $3 \leq n \leq k + 3$. Then one finds that

$$F_{k+4} = L_{k+2} + F_k = L_{k+2} + (L_{k-2} + L_{k-6} + \cdots + L_{k-2-4m} + e_k)$$

so that the theorem is true for $k + 4$, completing the induction step and the proof.

Also solved by David Zeitlin.

AN ALMOST LUCAS SUM

B-79 ***Proposed by Brother U. Alfred, St. Mary's College, Calif.***

Let $a = (1 + \sqrt{5})/2$. Determine a closed expression for

$$X_n = [a] + [a^2] + \cdots + [a^n] ,$$

where the square brackets mean "greatest integer in."

Solution by the Proposer.

If $b = (1 - \sqrt{5})/2$, $a^k = L_k - b^k$ with b negative and $|b| < 1$. Hence $[a^k] = L_k$ if k is odd and $[a^k] = L_k - 1$ if k is even. It follows that

$$X_n = \sum_{k=1}^{n} L_k - \left[\frac{n}{2}\right] = L_{n+3} - 3 - \left[\frac{n}{2}\right]$$

Also solved by J.L. Brown, Jr. and Jeremy C. Pond.

OUR MAN OF PISA

B-80 ***Proposed by Maxey Brooke, Sweeny, Texas.***

Solve the division alphametic

$$\begin{array}{r} \text{PISA} \\ \text{FIB}\,\overline{)\text{XONACCI}} \end{array}$$

where each letter represents one of the nine digits $1, 2, \cdots, 9$ and two letters may represent the same digit.

Solution by the Proposer.

$$\begin{array}{r} 9854 \\ 382\,\overline{)3764228} \end{array}$$

This solution may not be unique.

GAUSSIAN PRIMES

B-81 ***Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.***

Prove that only one of the Fibonacci numbers $1, 2, 3, 5, \cdots$ is a prime in the ring of Gaussian integers.

Solution by L. Carlitz, Duke University, Durham, N.C.

Since

$$F_{2n+1} = F_{n+1}^2 + F_n^2 = (F_{n+1} + F_n i)(F_{n+1} - F_n i) \quad ,$$

it follows that F_{2n+1} is composite in the Gaussian ring for all $n > 0$. Since $F_{2n} = F_n L_n$ it follows that F_{2n} is composite in the ring of integers (and therefore in the Gaussian ring) for $n > 2$. For $n = 2$, $F_4 = 3$; for $n = 1$, $F_2 = 1$.

Also solved by Sidney Kravitz and the proposer.

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