MORE FIBONACCI IDENTITIES

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In an earlier article [1] the author has discussed in detail the properties of a set of polynomials $B_n(x)$ and $b_n(x)$. It has been shown that [2],

$$B_n(x) = \left[\left\{(x + 2 + \sqrt{x^2 + 4x})/2\right\}^{n+1} - \left\{(x + 2 - \sqrt{(x^2 + 4x)}/2\right\}^{n+1}\right] \Big/ \sqrt{(x^2 + 4x)}$$

Putting $x = 1$ and simplifying we can show that

$$B_n(1) = F_{2n+2} \tag{1a}$$

where F_n is the n^{th} Fibonacci number.

Hence,

$$b_n(1) = B_n(1) - B_{n-1}(1) = F_{2n+2} - F_{2n} = F_{2n+1} \tag{1b}$$

We shall now use (1) and the properties of B_n and b_n to establish some interesting Fibonacci identities:

It has been shown that [1],

$$\begin{vmatrix} B_m & B_n \\ b_m & b_n \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{n-r} \\ b_{m-r-1} & b_{n-r-1} \end{vmatrix} \tag{2}$$

and

$$\begin{vmatrix} B_m & B_n \\ B_{m-1} & B_{n-1} \end{vmatrix} = \begin{vmatrix} B_{m-r} & B_{n-r} \\ B_{m-r-1} & B_{n-r-1} \end{vmatrix} \tag{3}$$

From (1) and (2) we can establish that

$$\begin{vmatrix} F_m & F_n \\ F_{m-1} & F_{n-1} \end{vmatrix} = \begin{vmatrix} F_{m-2r} & F_{n-2r} \\ F_{m-2r-1} & F_{n-2r-1} \end{vmatrix} \tag{4}$$

and from (1) and (3) that

$$(5)\qquad \begin{vmatrix} F_m & F_n \\ F_{m-2} & F_{n-2} \end{vmatrix} = \begin{vmatrix} F_{m-2r} & F_{n-2r} \\ F_{m-2r-2} & F_{n-2r-2} \end{vmatrix}$$

Using equations (33)-(37) of [1] we may deduce that [3],

$$(6)\qquad \begin{aligned} F_2 + F_6 + \cdots + F_{4n-2} &= F_{2n}^2 \\ F_1 + F_5 + \cdots + F_{4n-3} &= F_{2n-1}F_{2n} \\ F_3 + F_7 + \cdots + F_{4n-1} &= F_{2n}F_{2n+1} \\ F_4 + F_8 + \cdots + F_{4n} &= F_{2n}F_{2n+2} \end{aligned}$$

From (33)-(37) of [1] we may establish the identities

$$(x^2 + 4x)\sum_0^n B_r^2 = B_{2n+2} - (2n + 3)$$

$$(x^2 + 4x)\sum_0^n B_rB_{r+1} = B_{2n+3} - (n + 2)(x + 2)$$

$$(x^2 + 4x)\sum_0^n b_rB_r = b_{2n+2} + (n + 1)x - 1$$

and

$$(x^2 + 4x)\sum_0^n b_r^2 = B_{2n+1} + 2(n + 1)$$

From the above identities and (1) we can deduce that

$$(7)\qquad 5(F_2F_4 + F_4F_6 + \cdots + F_{2n-2}F_{2n}) = F_{4n} - 3n$$

$$(8)\qquad 5(F_1F_2 + F_3F_4 + \cdots + F_{2n-1}F_{2n}) = F_{4n+1} + (n - 1)$$

$$(9) \qquad 5(F_1^2 + F_3^2 + \cdots + F_{2n-1}^2) = F_{4n} + 2n$$

and

$$(10) \qquad 5(F_2^2 + F_4^2 + \cdots + F_{2n}^2) = F_{4n+2} - (2n + 1)$$

Combining the identities of (9) and (10) we get

$$(11) \qquad 5(F_1^2 + F_2^2 + \cdots + F_n^2) = F_{2n+2} + F_{2n} + (-1)^{n+1}$$

Also, we have the well-known identity,

$$(12) \qquad F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$$

Hence from (11) and (12) we get

$$(13) \qquad F_{2n+2} + F_{2n} - 5F_{n+1}F_n = (-1)^n$$

From (14) and (31) of reference [1] we have the results,

$$(14) \qquad B_r^2 - B_{r+1}B_{r-1} = 1$$

and

$$(15) \qquad b_rB_r - b_{r+1}B_{r-1} = 1$$

Therefore, $(B_r/b_{r+1}) - (B_{r-1}/b_r) = 1/(b_rb_{r+1})$. Hence,

$$(1/b_{n-1}b_n) + (1/b_{n-2}b_{n-1}) + \cdots + (1/b_1b_2) = (B_{n-1}/b_n) - (B_0/b_1)$$

Since $B_0 = b_0 = 1$, we may write this result as,

$$\sum_1^n (1/b_rb_{r-1}) = (B_{n-1}/b_n)$$

Therefore

$$(16) \qquad (B_n/b_n) = 1 + (B_{n-1}/b_n) = 1 + \sum_1^n (1/b_r b_{r-1})$$

Similarly starting with (14) we can establish that

$$(17) \qquad (b_n/B_n) = 1 - \sum_1^n (1/B_r B_{r-1})$$

Combining the identities (16) and (17) we have,

$$(18) \qquad \left\{1 + \sum_1^n (1/b_r b_{r-1})\right\}\left\{1 - \sum_1^n (1/B_r B_{r-1})\right\} = 1$$

Substituting (1) in (18) we derive an interesting result that

$$(19) \qquad \left\{1 + \sum_1^n \frac{1}{F_{2n-1}F_{2n+1}}\right\}\left\{1 - \sum_1^n \frac{1}{F_{2n}F_{2n+2}}\right\} = 1$$

Many other interesting Fibonacci identities may be established using the properties of B_n and b_n, and it is left to the reader to develop these identities.

REFERENCES

1. M. N. S. Swamy, "Properties of the Polynomials Defined by Morgan-Voyce," Fibonacci Quarterly, Vol. 4, No. 1, pp. 73-81.
2. S. L. Basin, "The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory," Mathematics Magazine, Vol. 36, March-April 1963, pp. 84-97.
3. K. Siler, "Fibonacci Summations," Fibonacci Quarterly, Vol. 1, October 1963, pp. 67-69.

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