ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-581 ***Proposed by José Luiz Díaz, Polytechnic University of Catalunya, Spain***

Let n be a positive integer. Prove that

(a) $F_n^{F_{n+1}} + F_{n+1}^{F_{n+2}} + F_{n+2}^{F_n} < F_n^{F_n} + F_{n+1}^{F_{n+1}} + F_{n+2}^{F_{n+2}}$,

(b) $F_n^{F_{n+1}} F_{n+1}^{F_{n+2}} F_{n+2}^{F_n} < F_n^{F_n} F_{n+1}^{F_{n+1}} F_{n+2}^{F_{n+2}}$.

H-582 ***Proposed by Ernst Herrmann, Siegburg, Germany***

a) Let A denote the set $\{2, 3, 5, 8, \ldots, F_{m+2}\}$ of m successive Fibonacci numbers, where $m \geq 4$. Prove that each real number x of the interval $I = [(F_{m+2} - 1)^{-1}, 1]$ has a series representation of the form

$$x = \sum_{i=1}^{\infty} \frac{1}{F_{k_1} F_{k_2} F_{k_3} \ldots F_{k_i}}, \tag{1}$$

where $F_{k_i} \in A$ for all $i \in N$.

b) It is impossible to change the assumption $m \geq 4$ into $m \geq 3$, that is, if $A = \{2, 3, 5\}$ and $I = [1/4, 1]$, then there are real numbers with no representation of the form (1), where $F_{k_i} \in A$. Find such a number.

SOLUTIONS

Inspiring

H-568 ***Proposed by N. Gauthier, Royal Military College of Canada, Kingston, Ontario (Vol. 38, no. 5, November 2000)***

The following was inspired by Paul. S. Bruckman's Problem B-871 in *The Fibonacci Quarterly* (proposed in Vol. 37, no. 1, February 1999; solved in Vol. 38, no. 1, February 2000).

"For integers $n, m \geq 1$, prove or disprove that

$$f_m(n) \equiv \frac{1}{\binom{2n}{n}^2} \sum_{k=0}^{2n} \binom{2n}{n}^2 |n-k|^{2m-1}$$

is the ratio of two polynomials with integer coefficients $f_m(n) = P_m(n)/Q_m(n)$, where $P_m(n)$ is of degree $\lfloor\frac{3m}{2}\rfloor$ in n and $Q_m(n)$ is of degree $\lfloor\frac{m}{2}\rfloor$; determine $P_m(n)$ and $Q_m(n)$ for $1 \le m \le 5$."

Solution by Paul S. Bruckman, Sacramento, CA

We let the combinatorial number read "a choose b" be denoted by the symbol ${}_aC_b$. After a bit of manipulation, we may express $f_m(n)$ as follows:

$$({}_{2n}C_n)^2 f_m(n) = 2\sum_{k=0}^{n}({}_{2n}C_k)^2(n-k)^{2m-1} = 2g_m(n), \text{ say.}$$

That is,

$$g_m(n) = \sum_{k=0}^{n}({}_{2n}C_k)^2(n-k)^{2m-1}. \tag{1}$$

For convenience, we make the following definitions:

$$A_r(n) = \sum_{k=0}^{n}({}_{2n}C_k)^2(n-k)^r; \tag{2}$$

$$B_r(n) = \sum_{k=0}^{n}({}_{2n+1}C_k)^2(n-k)^r; \tag{3}$$

$$U(n) = {}_{4n}C_{2n};\ V(n) = ({}_{2n}C_n)^2. \tag{4}$$

Note that $g_m(n) = A_{2m-1}(n)$ and $f_m(n) = 2A_{2m-1}(n)/V(n)$. Also note that

$$U(n-1) = n(2n-1)U(n)/\{2(4n-1)(4n-3)\},\ V(n-1) = n^2V(n)/\{4(2n-1)^2\}.$$

The following combinatorial identities are either directly found or easily derived from identities given in [1]; in some cases, their derivation is a bit lengthy, and is therefore abridged here:

$$A_0(n) = 1/2\{U(n)+V(n)\} \text{ (Identity (3.68) in [1]);} \tag{5}$$

$$B_0(n) = (4n+1)U(n)/(2n+1) \text{ (Identity (3.69) in [1]);} \tag{6}$$

$$A_2(n) = n^2U(n)/2(4n-1) \text{ (Identity (3.76) in [1] with } 2n \text{ replacing } n\text{);} \tag{7}$$

$$A_1(n) = nV(n)/4. \tag{8}$$

Proof of (8): The summand portion $(n-k)$ in $A_1(n)$ is equal to $(n^2-k^2+(n-k)^2)/2n$. Thus, after simplification,

$$\begin{aligned}2nA_1(n) &= n^2A_0(n) - 4n^2B_0(n-1) + A_2(n)\\ &= n^2(U(n)+V(n))/2 - 2n^3U(n)/(4n-1) + n^2U(n)/(4n-1)4n^2,\end{aligned}$$

which reduces to (8). □

$$\sum_{k=0}^{n}({}_{2n+1}C_k)^2(2n+1-2k)^2 = (2n+1)U(n) \text{ (Identity (3.76) in [1], with } 2n+1 \text{ replacing } n\text{).} \tag{9}$$

Now the summand portion $(n-k)$ in $B_1(n)$ may be written as

$$\{4n^2-1+(2n+1-2k)^2-4k^2\}/\{4(2n+1)\}.$$

It then follows that

$$4(2n+1)B_1(n) = (4n^2-1)B_0(n) + C(n) - 4(2n+1)^2(A_0(n) - V(n)),$$

where $C(n)$ is the expression given in the left member of (9). Then, after simplification, we obtain the following:

$$B_1(n) = (2n+1)V(n)/2 - (4n+1)U(n)/\{2(2n+1)\}. \tag{10}$$

Next, we note that

$$(n-k)^3 = (n-k)^2\{(2n-k)^2 - k^2\}/4n.$$

Then, using the above definitions, we see that

$$4nA_3(n) = 4n^2\{B_2(n-1) + 2B_1(n-1) + B_0(n-1)\} - 4n^2B_2(n-1);$$

hence,

$$A_3(n) = 2nB_1(n-1) + nB_0(n-1).$$

After further simplification, we obtain

$$A_3(n) = n^3V(n)/\{4(2n-1)\}. \tag{11}$$

From the definitions given in (1) and (2), along with the relation $f_m(n) = 2g_m(n)/V(n)$, and using the results of (8) and (11) we therefore have

$$f_1(n) = n/2, \quad f_2(n) = n^3/\{2(2n-1)\}. \tag{12}$$

We may prove Gauthier's conjecture by induction (on m). However, due to considerations of length, we can only outline the procedure and omit the details. The required tool for the proof is the following recurrence satisfied by the $f_m(n)$'s:

$$f_{m+2}(n) = 2n^2f_{m+1}(n) - n^4f_m(n) + n^4f_m(n-1). \tag{13}$$

Proof of (13):

$$V(n)\{f_{m+2}(n) - 2n^2f_{m+1}(n) + n^4f_m(n)\}/2$$
$$= \sum_{k=0}^{n}({}_{2n}C_k)^2(n-k)^{2m-1}\{(n-k)^4 - 2n^2(n-k)^2 + n^4\}.$$

Note that the quantity in braces is equal to $\{(n-k)^2 - n^2\}^2 = k^2(2n-k)^2$; therefore, the last summation may be expressed as follows:

$$(2n)^2(2n-1)^2\sum_{k=1}^{n}({}_{2n-2}C_{k-1})^2(n-k)^{2m-1}$$
$$= 4n^2(2n-1)^2\sum_{k=0}^{n-1}({}_{2n-2}C_k)^2(n-1-k)^{2m-1}$$
$$= 4n^2(2n-1)^2f_m(n-1)V(n-1)/2 = n^4f_m(n-1)V(n)/2,$$

which reduces to (13). □

Instead of applying induction directly on (13), we transform this recurrence and apply it to a modified set of functions. Namely, we make the following transformation:

$$k_m(n) = f_m(n)T_m(n), \tag{14}$$

where

$$T_m(n) = 2^{r+1}(n-1/2)^{(r)} = 2(2n-1)(2n-3)\dots(2n-2r+1) \tag{15}$$

and

$$r = [m/2]. \tag{16}$$

Therefore, the $T_m(n)$'s are polynomials in n of degree r. By making the substitution indicated in (14) into the recurrence (13), we obtain our modified recurrence relation. It becomes more convenient to dichotomize this relation into the two cases $m = 2r$ and $m = 2r+1$:

$$k_{2r+2}(n) - 2n^2(2n-1-2r)k_{2r+1}(n) + n^4(2n-1-2r)k_{2r}(n) = n^4(2n-1)k_{2r}(n-1), \tag{17}$$

$$k_{2r+3}(n) - 2n^2k_{2r+2}(n) + n^4(2n-1-2r)k_{2r+1}(n) = n^4(2n-1)k_{2r+1}(n-1). \tag{18}$$

From (12) and the relation in (14), we obtain the initial values

$$k_1(n) = n,\ k_2(n) = n^3,\ k_3(n) = n^4,\ k_4(n) = 3n^6 - 5n^5 + n^4. \tag{19}$$

It follows (by an easy induction) from (17), (18), and (19) that the $k_m(n)$'s are polynomials in n with integer coefficients.

The following results are posited:

$$k_{2r}(n) = a_{2r}n^{3r} + R_{3r-1}(n),\ k_{2r-1}(n) = a_{2r-1}n^{3r-2} + R_{3r-3}(n); \tag{20}$$

$$a_{2r} = (2r-1)!/2^{r-1},\ a_{2r-1} = (2r-2)!/2^{r-1}. \tag{21}$$

In these formulas, the functions $R_M(n)$ are polynomials in n of degree M. In order to prove (20) and (21), we must first verify that they yield the correct values for $r = 1$ and $r = 2$. Using (19), we find that $a_1 = a_2 = a_3 = 1$, and $a_4 = 3$, thereby validating (20)-(21) for $r = 1$ and $r = 2$. If we apply the recurrence (17) to find $k_{2r+2}(n)$, expand each expression using the putative expressions in (20)-(21), and compare coefficients, we find that the coefficients of n^{3r+5} and n^{3r+4} vanish, while the coefficient of n^{3r+3} is found by the first formula in (21) with $r+1$ replacing r. This establishes the first half of the inductive step.

Then applying (18) to obtain $k_{2r+3}(n)$ and repeating the process, we find that the coefficients of n^{3r+6} and n^{3r+5} vanish, while the coefficient of n^{3r+4} is found by the second formula in (21) with $r+1$ replacing r. This establishes the second half of the inductive step. This is essentially equivalent to Gauthier's conjecture, with the added bonus of an expression for the leading term of $k_m(n)$.

Note that the degree of $k_{2r}(n)$ is $3r$, while the degree of $k_{2r-1}(n)$ is $3r-2$; this fact may be expressed concisely as follows: the degree of $k_m(n)$ is $[3m/2]$.

Having established (20)-(21), we may then revert to the original definitions. That is, we may express $f_m(n)$ as the ratio $P_m(n)/Q_m(n)$ of two polynomials with integer coefficients, where

$$P_m(n) = k_m(n) \text{ and } Q_m(n) = 2^r(n-1/2)^{(r)} = 2(2n-1)(2n-3)\dots(2n-2r+1),$$

with $r = [m/2]$. Thus, the degree of $P_m(n)$ is $[3m/2] = m + r$, while the degree of $Q_m(n)$ is r. This completes the demonstration of Gauthier's conjecture.

It only remains to fulfill the last part of the problem, namely, to display the functions $f_m(n)$ for $m = 1, 2, 3, 4, 5$. Since we already know that $f_m(n) = P_m(n)/Q_m(n)$, where

$$Q_m(n) = 2(2n-1)(2n-3)\dots(2n-2r+1),$$

it suffices to display the first few values of $P_m(n)$. As we have already shown,

$$P_1(n) = n,\; P_2(n) = n^3,\; P_3(n) = n^4,\; P_4(n) = n^4(3n^2 - 5n + 1).$$

Continuing by means of (17) and (18), we find the following:

$$P_5(n) = n^4(6n^3 - 12n^2 + 6n - 1),$$
$$P_6(n) = n^4(30n^5 - 150n^4 + 252n^3 - 185n^2 + 65n - 9),$$
$$P_7(n) = n^4(90n^6 - 510n^5 + 1074n^4 - 1128n^3 + 650n^2 - 198n + 25),$$
$$P_8(n) = n^4(630n^8 - 6300n^7 + 24990n^6 - 52200n^5 + 64506n^4 - 49356n^3 + 23111n^2 - 6087n + 691),$$

etc.

By means of a little program names Derive, the author obtained the expanded expressions for $P_r(n)$ from $r = 1$ to $r = 15$. These are available upon request. It would be desirable to identify the "Gauthier polynomials" $P_r(n)$ with more familiar polynomials already appearing in the literature, whose properties may already be known.

Reference

1. H. W. Gould. *Combinatorial Identities*. Morgantown, W. Va., 1972.

A High Exponent

<u>H-569</u> ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 38, no. 5, November 2000)

Let $\tau(n)$ and $\sigma(n)$ denote, respectively, the number of divisors of the positive integer n and the sum of such divisors. Let $e_2(n)$ denote the highest exponent of 2 dividing n. Let p be any odd prime, and suppose $e_2(p+1) = h$. Prove the following for all odd positive integers a:

$$e_2(\sigma(p^a)) = e_2(\tau(p^a)) + h - 1. \tag{$*$}$$

Solution by H.-J. Seiffert, Berlin, Germany

If m and n are any positive integers, then

$$\begin{gathered} e_2(m) = 0 \text{ if } m \text{ is odd},\;\; e_2(mn) = e_2(m) + e_2(n), \\ e_2(m+n) = \min(e_2(m), e_2(n)) \text{ if } e_2(m) \neq e_2(n). \end{gathered} \tag{1}$$

Let h and q be positive integers such that q is odd and $2^h q - 1 > 1$. We consider the positive integers

$$A_{k,m} := \sum_{j=0}^{2^k m-1} (2^h q - 1)^j = \frac{(2^h q - 1)^{2^k m} - 1}{2^h q - 2},$$

where k is any positive integer and m any odd positive integer. First, we prove that

$$e_2(A_{1,m}) = h \text{ for all odd } m \in N. \tag{2}$$

Since $A_{1,1} = 2^h q$, this is true for $m = 1$. Suppose that (2) holds for the odd positive integer m. Using the easily verified equation

$$A_{1,m+2} = (2^h q - 1)^4 A_{1,m} + 2^{h+1} q(2^{2h-1} q^2 - 2^h q + 1),$$

from (1) and the induction hypothesis, we obtain $e_2(A_{1,m+2}) = h$, so that (2) is established by induction. Next, we prove that, if $m \in N$ is odd, then

$$e_2(A_{k,m}) = k + h - 1 \text{ for all } k \in N. \tag{3}$$

By (2), this is true for $k = 1$. Suppose that (3) holds for $k \in N$. We have

$$A_{k+1,m} = A_{k,m}((2^h q - 1)^{2^k m} + 1) = 2A_{k,m}((2^{h-1} q - 1)A_{k,m} + 1),$$

so that, by (1) and the induction hypothesis, $e_2(A_{k+1,m}) = e_2(A_{k,m}) + 1 = k + h$. This completes the induction proof of (3).

Let p, a, and h be as in the proposal. Then there exist positive integers k, q, and m such that q and m are both odd, $p = 2^h q - 1 > 1$, and $a = 2^k m - 1$. Noting that

$$\sigma(p^a) = 1 + p + \cdots + p^a = \frac{p^{a+1} - 1}{p - 1} = A_{k,m}$$

and

$$\tau(p^a) = a + 1 = 2^k m,$$

we see that the requested equation $(*)$ is an immediate consequence of (3).

Also solved by L. A. G. Dresel, D. Iannucci, H. Kwong, R. Martin, J. Spilkes, and the proposer.

EDITORIAL REQUEST
Please send in proposals!!!