SOLVING SOME GENERAL NONHOMOGENEOUS RECURRENCE RELATIONS OF ORDER *r* BY A LINEARIZATION METHOD AND AN APPLICATION TO POLYNOMIAL AND FACTORIAL POLYNOMIAL CASES

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1. INTRODUCTION

Let $a_0, \ldots, a_{r-1}$ $(r \geq 2, a_{r-1} \neq 0)$ be some fixed real (or complex) numbers and $\{C_n\}_{n\geq 0}$ be a sequence of real (or complex) numbers. Let $\{T_n\}_{n=0}^{+\infty}$ be the sequence defined by the following nonhomogeneous relation of order *r*,

$$T_{n+1} = a_0T_n + a_1T_{n-1} + \cdots + a_{r-1}T_{n-r+1} + C_n, \text{ for } n \geq r-1, \tag{1}$$

where $T_0, \ldots, T_{r-1}$ are specified by the initial conditions. In the sequel, we refer to these sequences as *sequences (1)*. The solutions $\{T_n\}_{n\geq 0}$ of (1) may be given as follows: $T_n = T_n^{(h)} + T_n^{(p)}$, where $\{T_n^{(h)}\}_{n\geq 0}$ is a solution of the homogeneous part of (1) and $\{T_n^{(p)}\}_{n\geq 0}$ is a particular solution of (1). If $C_n = \sum_{j=0}^{d}\beta_j C_n^j$, solutions $\{T_n\}_{n\geq 0}$ of (1) may be expressed as $T_n = \sum_{j=0}^{d}\beta_j T_n^j$, where $\{T_n^j\}_{n\geq 0}$ is a solution of (1) for $C_n = C_n^j$. Sequences (1) are studied in the case of C_n polynomial or factorial polynomial (see, e.g., [2], [3], [4], [5], [7], [12], and [8]).

The purpose of this paper is to study a linearization process of (1) when $C_n = V_n$, where $\{V_n\}_{n\geq 0}$ is an *m*-generalized Fibonacci sequence whose $V_0, \ldots, V_{m-1}$ are the initial terms and

$$V_{n+1} = b_0V_n + \cdots + b_{m-1}V_{n-m+1}, \text{ for } n \geq m-1, \tag{2}$$

where $b_0, \ldots, b_{m-1}$ $(m \geq 2, b_{m-1} \neq 0)$ are given fixed real (or complex) numbers. This process permits the construction of a method for solving (1). In the polynomial and factorial polynomial cases, our linearization process allows us to express well-known particular solutions, particularly Asveld's polynomials and factorial polynomials, in another form. Examples and discussion are given.

This paper is organized as follows: In Section 2 we study a Linearization Process of (1). In Section 3 we apply this process to polynomial and factorial polynomial cases. Section 4 provides a concluding discussion.

2. LINEARIZATION PROCESS FOR SEQUENCES (1)

In this section we suppose $C_n = V_n$ with $\{V_n\}_{n\geq 0}$ defined by (2), where we set $m = s$ and $\sigma_2 = \{\mu_0, \ldots, \mu_t\}$ the set of its characteristic roots whose multiplicities are, respectively, $p_0, \ldots, p_t$.

Expression (1) implies that $V_{n+1} = T_{n+1} - \sum_{j=0}^{r-1} a_j T_{n-j}$ for any $n \geq r-1$. Let $n \geq r+s-1$, then for any j $(0 \leq j \leq s-1)$ we have $V_{n-j} = T_{n-j} - \sum_{k=0}^{r-1} a_k T_{n-j-k-1}$. Then from (2) we derive that

$$T_{n+1} = \sum_{j=0}^{r-1} a_j T_{n-j} + \sum_{j=0}^{s-1} b_j T_{n-j} - \sum_{j=0}^{s-1} \sum_{k=0}^{r-1} b_j a_k T_{n-j-k-1}. \tag{3}$$

Expression (3) shows that T_{n+1} $(n \geq r+s-1)$ is a linear recurrence relation of order $r+s$; more precisely, we have

$$T_{n+1} = (a_0 + b_0)T_n + \sum_{j=0}^{r_1-1} (a_j + b_j - c_j)T_{n-j} + \sum_{j=r_1}^{r_2-1} v_j T_{n-j} - \sum_{j=r_2}^{r+s-1} c_j T_{n-j},$$

where $c_j = \sum_{k+p=j;\, k \geq 1,\, p \geq 0} b_{k-1} a_p$ and $r_1 = \min(r, s)$, $r_2 = \max(r, s)$ with $v_j = a_j - c_j$ for $r > s$, $v_j = b_j - c_j$ for $r < s$, and $v_j = 0$ for $r = s$. Hence, we have the following result.

Theorem 2.1 (Linearization Process): Let $\{T_n\}_{n \geq 0}$ be a sequence (1) and $\{V_n\}_{n \geq 0}$ be a sequence (2), where $m = s$. Suppose $C_n = V_n$, then $\{T_n\}_{n \geq 0}$ is a sequence (2), where $m = r+s$. More precisely, $\{T_n\}_{n \geq 0}$ is a sequence (2) whose initial terms are $T_0, \ldots, T_{r+s-1}$ and whose characteristic polynomial is $p(x) = p_1(x)p_2(x)$, where $p_1(x) = x^r - \sum_{j=0}^{r-1} a_j x^{r-j-1}$ is the characteristic polynomial of the homogeneous part of (1) and $p_2(x) = x^s - \sum_{j=0}^{s-1} b_j x^{s-j-1}$ is the characteristic polynomial of (2).

Let $\sigma_1 = \{\lambda_0, \ldots, \lambda_q\}$ be the set of characteristic roots of the homogeneous part of (1) whose multiplicities are $n_0, \ldots, n_q$, respectively. Then $\sigma = \{v, p(v) = 0\} = \sigma_1 \cup \sigma_2$. Set $\sigma = \{v_0, \ldots, v_k\}$, where $v_i = \mu_i$ for $0 \leq i \leq t$ and $v_{i+t} = \lambda_{i-1}$ for $1 \leq i \leq k-t+1$. If $\sigma_1 \cap \sigma_2 = \emptyset$, we have $k = q + t + 1$, and if not, $k = q+t+1-u$, where u is the cardinal of $\sigma_1 \cap \sigma_2$. In the latter case, the Linearization Process shows that the multiplicity of $v_j \in \sigma_1 \cap \sigma_2$ is $m_j = n_j + p_j$, where n_j and p_j are multiplicities of v_j in $p_1(x)$ and $p_2(x)$, respectively. Therefore, we derive the Binet formula of $\{T_n\}_{n \geq 0}$ as

$$T_n = \sum_{j=0}^{t} R_j(n) v_j^n + \sum_{j=1}^{k-t+1} R_{j+t}(n) v_{j+t}^n \tag{4}$$

with $R_j(n) = \sum_{i=0}^{m_j-1} \beta_{ji} n^i$, where m_j is the multiplicity of v_j in $p(x) = p_1(x)p_2(x)$ and β_{ji} are constants derived as solution of a linear system of $r+s$ equations (see, e.g., [9] and [11]).

Because v_j for $t+1 \leq j \leq k$ satisfies $p_1(v_j) = 0$, we show that the sequence $\{T_n^{(h)}\}_{n \geq 0}$ defined by $T_n^{(h)} = \sum_{j=1}^{k-t+1} R_{j+t}(n) v_{j+t}^n$ is a solution of the homogeneous part of (1). Thus, we have the following result.

Theorem 2.2: Let $\{T_n\}_{n \geq 0}$ be a sequence (1) and $\{V_n\}_{n \geq 0}$ a sequence (2), where $m = s$. Suppose $C_n = V_n$, then the sequence $\{T_n^{(p)}\}_{n \geq 0}$, defined by

$$T_n^{(p)} = T_n - T_n^{(h)} = \sum_{j=0}^{t} R_j(n) v_j^n,$$

is a particular solution of (1).

Suppose $v_0 = \mu_0 = 1 \in \sigma_2$, then Binet's formula implies that $V_n = Q_0(n) + \sum_{j=1}^{t} Q_j(n) \mu_j^n$, where $Q_j(n)$ are polynomials in n of degree $p_j -$. Then a solution $\{T_n^{(p)}\}_{n \geq 0}$ of (1) may be expressed as follows: $T_n^{(p)} = T_n^1 + T_n^2$, where $\{T_n^1\}_{n\geq 0}$ and $\{T_n^2\}_{n\geq 0}$ are the solutions of (1) for, $C_n = Q_0(n)$ and $C_n = \sum_{j=1}^t Q_j(n)\mu_j^n$, respectively. We call $\{T_n^1\}_{n\geq 0}$ the *polynomial solutions of (1)*, corresponding to the polynomial part of C_n.

Corollary 2.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) and $\{V_n\}_{n\geq 0}$ a sequence (2). Suppose $\nu_0 = \mu_0 = 1 \in \sigma_2$. Then the polynomial solution $\{T_n^1\}_{n\geq 0}$ of (1) is given by $T_n^1 = R_0(n)$, where $R_0(n) = \sum_{i=0}^{m_0-1}\beta_{0i}n^i$ is derived from the Binet formula (4) of the linearized expression (3) of $\{T_n\}_{n\geq 0}$. More precisely:

(a) If $1 \notin \sigma_1$, we have $T_n^1 = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1$, where $m_0 = p_0$ is the multiplicity of $\mu_0 = 1$ in $p_2(x)$.

(b) If $1 \in \sigma_1$, we have $T_n^1 = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = n_0 + p_0 - 1$, where n_0 and p_0 are multiplicities of $\lambda_0 = \mu_0 = 1$ in $p_1(x)$ and $p_2(x)$, respectively.

Corollary 2.1 shows that the polynomial solution $\{T_n^1\}_{n\geq 0}$ of (1) is nothing but the polynomial part of (4), corresponding to the solution of (1) for C_n, equal to the polynomial part in the Binet decomposition of $\{V_n\}_{n\geq 0}$.

Example 2.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) whose initial terms are T_0, T_1, and $T_{n+1} = a_0T_n + a_1T_{n-1} + V_n$ for $n \geq 1$, where $\{V_n\}_{n\geq 0}$ is a sequence (2) with $m = s$. Then the Linearization Process implies that $\{T_n\}_{n\geq 0}$ is a sequence (2), where $m = s+2$, whose initial terms are $T_0, \ldots, T_{s+1}$ and whose coefficients are $u_0 = a_0 + b_0$, $u_1 = a_1 + b_1 - a_0b_0$, $u_2 = b_1 - a_0b_1 - a_1b_0, \ldots, u_{s-1} = b_{s-1} - a_0b_{s-2} - a_1b_{s-3}$, $u_s = a_0b_{s-1} - a_1b_{s-2}$, and $u_{s+1} = -a_1b_{s-1}$.

3. APPLICATIONS TO POLYNOMIAL AND FACTORIAL POLYNOMIAL CASES

3.1 Polynomial Case

In this subsection we consider $C_n = \sum_{j=0}^d \beta_j n^j$, where $n \in \mathbb{N}$. Let us first connect this case with the situation of Section 2. To this aim, we can show easily that if $\{V_n\}_{n\geq 0}$ is a sequence such that $V_n = \sum_{j=0}^d \beta_j n^j$, for $n \geq 0$, then $\{V_n\}_{n\geq 0}$ is a sequence (2) with $m = d+1$ whose initial terms are $V_0, \ldots, V_d$ and coefficients $b_j = (-1)^j\binom{d-j}{d+1}$, where $\binom{k}{n} = \frac{n!}{k!(n-k)!}$, are derived from its characteristic polynomial $p_2(x) = (x-1)^{d+1}$. Particularly, for $C_n = n^j$, we derive the following proposition from Corollary 2.1.

Proposition 3.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) and let $C_n = n^j$. Then the polynomial solution $\{P_j(n)\}_{n\geq 0}$ of (1) is given by $P_j(n) = R_0(n)$, where $R_0(n) = \sum_{i=0}^{m_0-1}\beta_{0i}n^i$ is derived from the Binet formula (4) of the linearized expression (3) of $\{T_n\}_{n\geq 0}$. More precisely:

(a) If $1 \notin \sigma_1$, we have $P_j(n) = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = j$.

(b) If $1 \in \sigma_1$, we have $P_j(n) = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = n_0 + j$, where n_0 is the multiplicity of $\lambda_0 = 1$ in $p_1(x)$.

More generally, we have the following result.

Proposition 3.2: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) and let $C_n = \sum_{j=0}^d \beta_j n^j$. Then the polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is $P(n) = R_0(n) = \sum_{j=0}^d \beta_j P_j(n)$, where $R_0(n) = \sum_{i=0}^{m_0-1}\beta_{0i}n^i$ is derived from the Binet formula (4) of the linearized expression (3) or $\{T_n\}_{n\geq 0}$. More precisely:

(a) If $1 \notin \sigma_1$, we have $P(n) = R_0(n)$ with $R_0(x)$ of degree d.

(b) If $1 \in \sigma_1$, we have $P(n) = R_0(n)$ with $R_0(x)$ of degree $m_0 - 1 = n_0 + d$, where n_0 is the multiplicity of $\lambda_0 = 1$ in $p_1(x)$.

Propositions 3.1 and 3.2 show that particular polynomial solutions $P_j(n)$ $(0 \le j \le d)$ are the well-known Asveld polynomials studied in [3], [5], [8], and [12]. Our method of obtaining $P_j(n)$ $(0 \le j \le d)$ is different. For their computation, we applied the *Linearization Process* of Section 2 to $\{T_n\}_{n \ge 0}$. Thus, the Binet formula (4) of the linearized expression (3) of (1) allows us to conclude that $P_j(n)$ can be considered as a polynomial part of (4). For $\lambda_0 = 1 \in \sigma_1$, we have $m_0 \ge j+2$, and Proposition 3.1 shows that $P_j(n)$ may be of degree $\ge j+1$ because the α_{0i} are not necessarily null for $j+1 \le i \le m_0 - 1$. This result has been verified by the authors with the aid of another method devised for solving equations (1) for a general C_n.

3.2 Factorial Polynomial Case

In this subsection, let $C_n = \sum_{j=0}^{d} \beta_j n^{(j)}$, where $n^{(j)} = n(n-1)\cdots(n-j+1)$. Note that $n^{(j)} = j!\binom{n}{j}$ for $j \ge 1$ and $n^{(0)} = 1$ $(0^{(0)} = 1)$. This case is related to the situation of Section 2 as follows. Consider Stirling numbers of the first kind $s(t, j)$ and Stirling numbers of the second kind, $S(t, j)$, which are defined by $x^{(j)} = \sum_{t=0}^{j} s(t, j)x^t$ and $x^i = \sum_{t=0}^{i} S(t, i)x^{(t)}$ (see, e.g., [1], [6], [7], and [10]). Hence, for any $j \ge 1$, we have $n^{(j)} = \sum_{t=0}^{j} s(t, j)n^t$. Therefore, $\{n^{(j)}\}_{n \ge 0}$ is a sequence (2), where $s = j+1$. We then derive the following proposition from Proposition 3.2.

Proposition 3.3: Let $\{T_n\}_{n \ge 0}$ be a sequence (1) and $C_n = n^{(j)}$. Then the factorial polynomial solution $\{\widetilde{P}_j(n)\}_{n \ge 0}$ of (1) is given by $\widetilde{P}_j(n) = \widetilde{R}_{0,j}(n)$, where $\widetilde{R}_{0,j}(n) = \sum_{t=0}^{j} s(j, t)P_t(n)$ with $P_t(n) = \sum_{i=0}^{t+m_0-1} \alpha_{0i}^t n^i$ $(0 \le t \le j)$ are solutions of the linearized expression (3) of $\{T_n\}_{n \ge 0}$ for $C_n = n^t$ $(0 \le t \le j)$. More precisely:

(a) If $1 \notin \sigma_1$, we have $\widetilde{P}_j(n) = \sum_{q=0}^{j}(\sum_{t=q}^{j} s(j, t)\gamma_{tq})n^{(q)}$, where $\gamma_{tq} = \sum_{i=q}^{t} \alpha_{0i}^t S(i, q)$, with $S(i, q)$ the Stirling numbers of the second kind.

(b) If $1 \in \sigma_1$, we have $\widetilde{P}_j(n) = \sum_{q=0}^{j+m_0-1}(\sum_{t=q}^{j+m_0-1} s(j, t)\gamma_{tq})n^{(q)}$, where $\gamma_{tq} = \sum_{i=q}^{t+m_0-1} \alpha_{0i}^t S(i, q)$, with $m_0 \ge 1$ the multiplicity of $\lambda_0 = 1$.

More generally, we have the following proposition.

Proposition 3.4: Let $\{T_n\}_{n \ge 0}$ be a sequence (1) and $C_n = \sum_{j=0}^{d} \beta_j n^{(j)}$. Then, the factorial polynomial solution $\{\widetilde{P}(n)\}_{n \ge 0}$ of (1) is given by $P(n) = R_0(n) = \sum_{j=0}^{d} \beta_j \widetilde{P}_j(n)$, where $\widetilde{P}_j(n)$ are factorial polynomial solutions of (1) for $C_n = n^{(j)}$ given by Proposition 3.3.

Propositions 3.3 and 3.4 show that particular factorial polynomial solutions $\widetilde{P}_j(n)$ $(0 \le j \le d)$ are the well-known Asveld factorial polynomials studied in [5] and [7]. Our method of obtaining $\widetilde{P}_j(n)$ $(0 \le j \le d)$ is different from those above. As for the polynomial case, if $1 \in \sigma_1$, we can show that $\widetilde{P}_j(n)$ $(0 \le j \le d)$ may be of *degree* $\ge j+1$. This result has also been verified by the authors using another method for solving (1) in the general case.

Example 3.1: Let $\{T_n\}_{n \ge 0}$ be a sequence (1) whose initial conditions are T_0, T_1, and $T_{n+1} = 3T_n - 2T_{n-1} + V_n$ for $n \ge 1$, where $V_n = n$. It is easy to see that $V_{n+1} = 2V_n - V_{n-1}$; therefore, the Linearization Process of Section 2 and Example 2.1 imply that $T_{n+1} = 5T_n - 9T_{n-1} + 7T_{n-2} - 2T_{n-3}$ for $n \ge 3$, where the initial conditions are T_0, T_1, $T_2 = 3T_1 - 2T_0 + 1$, and $T_3 = 7T_1 - 6T_0 + 5$. The characteristic polynomial of $\{T_n\}_{n\geq 0}$ is $p(x) = (x-1)^3(x-2)$. So the Binet formula of $\{T_n\}_{n\geq 0}$ is $T_n = P(n) + \eta 2^n$ for any $n \geq 0$, where $P(n) = an^2 + bn + c$. Also, the coefficients a, b, c, and η are a solution of the linear system of 4 equations, (S): $P(n) + \eta 2^n = T_n$, $n = 0, 1, 2, 3$. A straight computation allows us to verify that (S) is a Cramer system which owns a unique solution a, b, c, and η. In particular, we have $a = -\frac{1}{2}$. Hence, the polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is of degree 2.

4. CONCLUDING DISCUSSION AND EXAMPLE

4.1 Method of Substitution and Linearization Process

For $C_n = \sum_{j=0}^{d} \beta_j n^j$ (or $C_n = \sum_{j=0}^{d} \beta_j n^{(j)}$), the usual way for searching the particular polynomial (or factorial polynomial) solutions $\{P(n)\}_{n\geq 0}$ (or $\{\widetilde{P}(n)\}_{n\geq 0}$) of (1), and hence the Asveld polynomials (or factorial polynomials), is to consider them in the following form:

$$P(n) = \sum_{j=0}^{d} A_j n^j, \quad \widetilde{P}(n) = \sum_{j=0}^{d} A_j n^{(j)}. \tag{5}$$

Then the coefficients A_j $(0 \leq j \leq d)$ are computed from a series of equations that are obtained from the *substitution* of (5) in (1) (see, e.g., [3], [4], [5], [7], [8], and [12]).

The natural question is: How can we compare the Linearization Process of Section 2 and the method of substitution for searching particular solutions of (1) in polynomial and factorial polynomial cases? The Linearization Process of Section 2 shows that:

(a) If $\lambda_0 = 1 \notin \sigma_1$ [i.e., 1 is not a characteristic root of the homogeneous part of (1)], the Linearization Process shows that $\{P(n)\}_{n\geq 0}$ (or $\{\widetilde{P}(n)\}_{n\geq 0}$) is of the form (5). And the coefficients A_j $(0 \leq j \leq d)$ of (5) are obtained with the aid of the Binet formula applied directly to the linearized expression (3) of (1).

(b) If $\lambda_0 = 1 \in \sigma_1$ [i.e., 1 is a characteristic root of the homogeneous part of (1)], then these solutions may be of degree $\geq d$. More precisely, we have $P(n) = \sum_{j=0}^{d+n_0} A_j n^j$ and $\widetilde{P}(n) = \sum_{j=0}^{d+n_0} A_j n^{(j)}$, where n_0 is the multiplicity of $\lambda_0 = 1 \in \sigma_1$. If $P(n)$, or $\widetilde{P}(n)$, is of degree d, we must have $A_j = 0$ for $d+1 \leq j \leq d+n_0$. This means that we have some constraints on the coefficients $a_0, \ldots, a_{r-1}$, or on the initial terms $T_0, \ldots, T_{r-1}$.

The following simple example helps to make precise the difference between the Linearization Process and the method of substitution.

Example 4.1: Let $\{T_n\}_{n\geq 0}$ be a sequence (1) whose initial terms are T_0, T_1, and $T_{n+1} = a_0 T_n + a_1 T_{n-1} + V_n$ for $n \geq 1$, where $a_0 = 1 - \alpha$, $a_1 = \alpha$ with $\alpha \neq 1$, and $V_n = n$. Then we can see that $V_{n+1} = 2V_n - V_{n-1}$. Hence, the Linearization Process of Section 2 implies that

$$T_{n+1} = (3-\alpha)T_n + 3(\alpha-1)T_{n-1} - (3\alpha-1)T_{n-2} + \alpha T_{n-3} \text{ for } n \geq 3,$$

where initial terms are T_0, T_1, $T_2 = (1-\alpha)T_1 + \alpha T_0 + 1$, and $T_3 = (\alpha^2 - \alpha + 1)T_1 + \alpha(1-\alpha)T_0 + (3-\alpha)$. The characteristic polynomial of $\{T_n\}_{n\geq 0}$ is $p(x) = (x-1)^3(x+\alpha)$, and its Binet formula is $T_n = P(n) + \eta\lambda_1^n$ for $n \geq 0$, where $P(n) = an^2 + bn + c$ and $\lambda_1 = -\alpha$. The coefficients a, b, c, and η are derived from the following linear system 4 equations (S): $P(2) + \eta\lambda_1^j = T_j$, $j = 0, 1, 2$, and 3. A straight computation allows us to see that (S) is a Cramer system which owns a unique solution a, b, c, and η if $\Delta_\alpha = 2\alpha^3 + 6\alpha^2 + 6\alpha - 2 \neq 0$. In particular, we have $a = (\alpha+1)^2 / \Delta_\alpha$. Therefore, the polynomial solution $\{P(n)\}_{n\geq 0}$ of (1) is of degree 1 if $\alpha = -1$, and of degree 2 if not.

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