CONSECUTIVE BINOMIAL COEFFICIENTS IN PYTHAGOREAN TRIPLES AND SQUARES IN THE FIBONACCI SEQUENCE

Florian Luca
Instituto de Matematicas de la UNAM, Campus Morelia, Apartado Postal 61-3 (Xangari)
CP 58 089, Morelia, Michoacan, Mexico
(*Submitted December 1999-Final Revision May 2000*)

In this note, we find all triples consisting of consecutive binomial coefficients

$$\binom{n}{k}\ \binom{n}{k+1}\ \binom{n}{k+2} \tag{1}$$

forming Pythagorean triples. The result is

Theorem: If the three numbers listed at (1) above form a Pythagorean triple, then $n=62$ and $k=26$ or 34.

We first notice that it is enough to assume that $k+2 \le n/2$. Indeed, if $k \ge n/2$, then one can use the symmetry of the Pascal triangle to reduce the problem to the previous one, while the case in which $k<n/2$ but $k+2>n/2$ is impossible because these conditions will lead to isosceles Pythagorean triangles which, as we all know, do not exist.

Proof: After performing the cancellations in the following equation,

$$\binom{n}{k}^2+\binom{n}{k+1}^2=\binom{n}{k+2}^2, \tag{2}$$

we get

$$(k+2)^2((k+1)^2+(n-k)^2)=(n-k)^2(n-k-1)^2. \tag{3}$$

We make the substitution $x:=n-k$ and $y:=k+1$. Equation (3) becomes

$$(y+1)^2(x^2+y^2)=x^2(x-1)^2. \tag{4}$$

Notice that equation (4) implies that x^2+y^2 is a square. Let $d:=\gcd(x,y)$.

We distinguish two cases:

Case 1.

$$\begin{cases} x=2duv, \\ y=d(u^2-v^2), \end{cases} \quad \text{where } \gcd(u,v)=1 \text{ and } u\not\equiv v \pmod 2. \tag{5}$$

Combining formulas (5) and equation (4), we get

$$(d(u^2-v^2)+1)(u^2+v^2)=2uv(2duv-1). \tag{6}$$

Since $\gcd(u^2+v^2, 2uv)=1$, it follows from equation (6) that $(u^2+v^2)\mid(2duv-1)$. Hence,

$$\frac{2duv-1}{u^2+v^2}=\frac{d(u^2-v^2)+1}{2uv}=d_1, \tag{7}$$

where d_1 is an integer. One can rewrite the two equations (7) as

$$\begin{cases} d(2uv)-d_1(u^2+v^2)=1, \\ d(u^2-v^2)-d_1(2uv)=-1. \end{cases} \tag{8}$$

One can now regard (8) as a linear system in two unknowns, namely, d and d_1. After solving it by using Kramer's rule, one gets

$$\begin{cases} d = \dfrac{-(u+v)^2}{u^4 - v^4 - 4u^2v^2}, \\ d_1 = \dfrac{-u^2 + v^2 - 2uv}{u^4 - v^4 - 4u^2v^2}. \end{cases} \tag{9}$$

Let $\Delta = u^4 - v^4 - 4u^2v^2$ be the determinant of the coefficient matrix. We now show that $\Delta = \pm 1$. Indeed, notice that since $u \not\equiv v \pmod 2$, it follows that Δ is odd. Assume that $|\Delta| > 1$ and let p be an odd prime divisor of Δ. From the first formula (9) and the fact that d is an integer, we get that $p \mid (u+v)$. Since $p \mid \Delta = u^4 - v^4 - 4u^2v^2 = (u+v)(u-v)(u^2+v^2) - 4u^2v^2$, it follows that $p \mid uv$. But since $p \mid (u+v)$ also, we get that $p \mid \gcd(u, v)$, which is impossible. Hence,

$$u^4 - v^4 - 4u^2v^2 = \pm 1. \tag{10}$$

Notice that equation (10) can be rewritten as $(2(u^2 - 2v^2))^2 - 5(2v^2)^2 = \pm 4$. It is well known that all positive integer solutions of $X^2 - 5Y^2 = \pm 4$ are of the form $X = L_t$ and $Y = F_t$ for some positive integer t, where $(L_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$ are the Lucas and the Fibonacci sequence, respectively, given by $L_0 = 2$, $L_1 = 1$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ and $F_{n+2} = F_{n+1} + F_n$, respectively.* Now equation (11) implies that

$$\begin{cases} F_t = 2v^2, \\ L_t = \pm 2(u^2 - 2v^2). \end{cases} \tag{12}$$

It is known (see, e.g., [3]) that the only Fibonacci numbers which are twice times a square are $F_0 = 0$, $F_3 = 2$, and $F_6 = 8$. Hence, for our case, we get $t = 3$, $v = 1$, and $t = 6$, $v = 2$, respectively. In the first case, we get $u = 2$. From formula (9), we get $d = 9$, and then from formulas (5), we get $x = 36$ and $y = 27$. This gives the solution $n = 62$ and $k = 26$, and by the symmetry of the Pascal triangle, $k = 34$ as well. The case $t = 6$ and $v = 2$ does not lead to an integer solution for u.

Case 2.

$$\begin{cases} x = d(u^2 - v^2), \\ y = 2duv, \end{cases} \quad \text{where } \gcd(u, v) = 1 \text{ and } u \not\equiv v \pmod 2. \tag{13}$$

This case is very similar to the preceding one. With the notations (13), equation (4) becomes

$$(d(2uv) + 1)(u^2 + v^2) = (u^2 - v^2)(d(u^2 - v^2) - 1). \tag{14}$$

Since $\gcd(u^2 + v^2, u^2 - v^2) = 1$, it follows that $(u^2 + v^2) \mid (d(u^2 - v^2) - 1)$. Hence, equation (14) implies that

$$\frac{d(2uv) + 1}{u^2 - v^2} = \frac{d(u^2 - v^2) - 1}{u^2 + v^2} = d_1, \tag{15}$$

where d_1 is an integer. One may now rewrite equation (15) as

---

* I could not find a reference for this fact.

$$\begin{cases} d(2uv) - d_1(u^2 - v^2) = -1, \\ d(u^2 - v^2) - d_1(u^2 + v^2) = 1. \end{cases} \tag{16}$$

Solving system (16) in terms of d and d_1 versus u and v, we get

$$\begin{cases} d = \dfrac{2u^2}{(u^2 - v^2)^2 - 2uv(u^2 + v^2)}, \\ d_1 = \dfrac{2uv + u^2 - v^2}{(u^2 - v^2)^2 - 2uv(u^2 + v^2)}. \end{cases} \tag{17}$$

One may again argue as in the preceding case that

$$(u^2 - v^2)^2 - 2uv(u^2 + v^2) = \pm 1. \tag{18}$$

Rewrite (18) as

$$(2(u^2 + v^2 - uv))^2 - 5(2uv)^2 = \pm 4. \tag{19}$$

Equation (19) implies that there exists $t > 0$ such that

$$\begin{cases} F_t = 2uv, \\ L_t = 2(u^2 + v^2 - uv). \end{cases} \tag{20}$$

Formulas (20) imply that

$$\frac{L_t - F_t}{2} = (u - v)^2. \tag{21}$$

Using the well-known fact that $L_t = F_t + 2F_{t-1}$ for all $t \geq 1$, it follows by formula (21) that

$$F_{t-1} = (u - v)^2. \tag{22}$$

It is well known (see [1] or [2]) that the only squares in the Fibonacci sequence are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, and $F_{12} = 144$. Hence, by formula (22), we get that $t = 1, 2, 3, 13$. None of these values gives integer solutions u, v from the system of equations (20). The Theorem is therefore proved.

ACKNOWLEDGMENT

I would like to thank an anonymous referee for suggestions that improved the quality of this paper. This note was written when I visited the Mathematical Institute of the Czech Academy of Sciences from September 1999 to August 2000. I would like to thank Michal Krízek and Larry Somer for interesting conversations regarding the Fibonacci sequence and the Mathematical Institute of the Czech Academy of Sciences for its hospitality.

REFERENCES

1. J. H. E. Cohn. "On Square Fibonacci Numbers." *J. London Math. Soc.* **39** (1964):537-40.
2. J. H. E. Cohn. "Squares in Some Recurrent Sequences." *Pacific J. Math.* **41** (1972):631-46.
3. P. Ribenboim & W. L. McDaniel. "The Square Terms in Lucas Sequences." *J. Number Theory* **58** (1996):104-22.

AMS Classification Numbers: 11B39, 11D25

❖❖❖