ON FIBONACCI AND PELL NUMBERS OF THE FORM kx^2
(Almost Every Term Has a $4r + 1$ Prime Factor)

Wayne L. McDaniel

University of Missouri-St. Louis, St. Louis, MO 63121

e-mail: mcdaniel@arch.umsl.edu

(Submitted October 1999-Final Revision June 2000)

1. INTRODUCTION

In 1983 and 1984, Neville Robbins showed that neither the Fibonacci nor the Pell number sequence has terms of the form px^2 for prime $p \equiv 3 \pmod 4$, with one exception in each sequence [3], [4]. The main idea of Robbins' paper can be used to prove a stronger result, namely, that with a small number of exceptions, neither sequence has terms of the form kx^2 if k is an integer all of whose prime factors are congruent to 3 modulo 4. An interesting corollary is that, with 11 exceptions, every term of the Fibonacci sequence has a prime factor of the form $4r+1$ and, similarly, with 5 exceptions, for the Pell sequence.

The solutions of $F_n = x^2$ and $F_n = 2x^2$ were found by Cohn [1], and of $F_n = kx^2$, for certain values of $k > 2$, by Robbins [5]; of particular interest is Robbins' result that there are 15 values of k, $2 < k \le 1000$, for which solutions exist, and he gives these solutions. We refer the reader to [5].

2. SOME IDENTITIES AND RESULTS

We shall use the following identities and well-known facts relating the Fibonacci and Lucas numbers:

$$F_{2n} = F_n L_n, \tag{1}$$

$$\gcd(F_n, L_n) = 2 \text{ if } 3 \mid n \text{ and } 1 \text{ otherwise}, \tag{2}$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2. \tag{3}$$

Let $S = \{3, 4, 6, 8, 16, 24, 32, 48\}$ and let $T = \{k' \mid k' > 1$ is square-free and each odd prime factor of k' is $\equiv 3 \pmod 4\}$. It may be noted that, in the following theorem, there is no loss of generality in assuming that k is square-free.

Theorem 1: If $n > 1$, then $F_n = kx^2$ for some square-free integer $k \ge 2$ whose odd prime factors are all $\equiv 3 \pmod 4$ iff $n \in S$.

Proof: The sufficiency:

$$\begin{array}{llll}
F_3 = 2 & (k=2) & F_{16} = 3\cdot 7\cdot 47 & (k = 987) \\
F_4 = 3 & (k=3) & F_{24} = 2^5\cdot 3^2\cdot 7\cdot 23 & (k = 322) \\
F_6 = 8 & (k=2) & F_{32} = 3\cdot 7\cdot 47\cdot 2207 & (k = 2178309) \\
F_8 = 3\cdot 7 & (k=21) & F_{48} = 2^6\cdot 3^2\cdot 7\cdot 23\cdot 47\cdot 1103 & (k = 8346401)
\end{array}$$

The necessity: Assume there exists at least one integer $n > 1$, $n \notin S$, such that F_n has the form $k'X^2$ for some $k' \in T$ and integer X. Then there exists a least such integer N; we let $F_N = kx^2$ for some $k \in T$ and integer x. Now N is not odd, since, by (3), if N is odd, then F_N is the sum of 2 squares and it is well known that the square-free part of the sum of 2 squares does not have a factor $\equiv 3 \pmod 4$. Let $N = 2m$. Then, by (1) and (2), $F_N = kx^2$ implies that there exist integers y and z, $x = yz$, such that either

(a) $F_m = y^2$ and $L_m = kz^2$,
(b) $F_m = 2y^2$ and $L_m = 2kz^2$,
(c) $F_m = k_1y^2$ and $L_m = k_2z^2$, or
(d) $F_m = 2k_1y^2$ and $L_m = 2k_2z^2$,

where $k_1k_2 = k$, $k_1 > 2$.

If (a), then by [1], $m = 1, 2$, or 12. But then $N = 2$, which is not possible since $F_2 = 1$, or $N = 4$ or 24, contrary to our assumption that $N \notin S$.

If (b), then by [2], $m = 3$ or 6, but then $N = 6$, contrary to our assumption, or $N = 12$, but $F_{12} \neq kx^2$.

If (c), then, since $m < N$ and $k_1 \in T$, $m = 4, 6, 8, 16, 24, 32$, or 48; that is, $N = 8$, 12, 16, 32, 48, 64, or 96. But 8, 16, 32, 48 are in S, $F_{12} \neq k_1x^2$, $4481 \mid F_{64}$, and $769 \mid F_{96}$ ($4481, 769 \equiv 1 \pmod 4$)).

If (d), then either $2k_1 \in T$ or, if k_1 is even, $k_1 = 2k_3$ and $F_m = 2k_1y^2 = k_3(2y^2)$, with $k_3 \in T$; hence, the argument of (c) applies with k_1 replaced by $2k_1$ or k_3.

It follows that, if $n \notin S$, then $F_n \neq k'x^2$ for any $k' \in T$.

Since $F_n \neq kx^2$ implies $F_n \neq k$, we immediately have

Theorem 2: If $n \neq 0, 1, 2$ or an element of S, then F_n has at least one prime factor of the form $4r + 1$.

If P_n denotes the n^{th} Pell number, and R_n the n^{th} term of the "associated Pell sequence" ($R_0 = 2, R_1 = 1$), then, with one minor change, properties (1), (2), and (3) hold: $P_{2m} = P_mR_m$, $\gcd(P_m, R_m) = 2$ if m is even and 1 otherwise, and $P_{2m+1} = p_m^2 + p_{m+1}^2$.

We have the following results for Pell numbers. The proofs require the known facts that P_n is a square iff $n = 1$ or 7 and P_n is twice a square iff $n = 1$ (see [4]); since the proofs parallel those of Theorems 1 and 2, we omit them.

Theorem 3: If $n > 1$, then $P_n = kx^2$ for some square-free integer k whose odd prime factors are all $\equiv 3 \pmod 4$ iff $n = 2, 4$, or 14.

Theorem 4: If $n \neq 0, 1, 2, 4$, or 14, then P_n has at least one prime factor of the form $4r + 1$.

REFERENCES

1. J. H. E. Cohn. "On Square Fibonacci Numbers." *J. London Math. Soc.* **39** (1964):537-41.
2. J. H. E. Cohn. "Eight Diophantine Equations." *Proc. London Math. Soc.* **16.3** (1966):153-66.
3. N. Robbins. "On Fibonacci Numbers of the Form PX^2, Where P Is Prime." *The Fibonacci Quarterly* **21.3** (1983):266-71.
4. N. Robbins. "On Pell Numbers of the Form PX^2, Where P Is Prime." *The Fibonacci Quarterly* **22.4** (1984):340-48.
5. N. Robbins. "Fibonacci Numbers of the form cx^2, Where $1 \leq c \leq 1000$." *The Fibonacci Quarterly* **28.4** (1990):306-15.

AMS Classification Number: 11B39