AN INTERESTING PROPERTY OF A RECURRENCE RELATED TO THE FIBONACCI SEQUENCE

M. N. Deshpande
Institute of Science, Nagpur-440001, India
E-mail: dpratap@nagpur.dot.net

Andrej Dujella
Dept. of Mathematics, University of Zagreb, Bijenicka cesta 30, 10000 Zagreb, Croatia
E-mail: duje@math.hr

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1. INTRODUCTION

The sequence of Fibonacci numbers with even subscripts (F_{2n}) has one remarkable property. If we choose three successive elements of this sequence, then the product of any two of them increased by 1 is a perfect square. Indeed,

$$F_{2n} \cdot F_{2n+2} + 1 = F_{2n+1}^2, \quad F_{2n} \cdot F_{2n+4} + 1 = F_{2n+2}^2.$$

This property was studied and generalized by several authors (see references). Let us just mention that Hoggatt and Bergum [8] proved that the number $d = 4F_{2n+1}F_{2n+2}F_{2n+3}$ has the property that $F_{2n} \cdot d + 1$, $F_{2n+2} \cdot d + 1$, and $F_{2n+4} \cdot d + 1$ are perfect squares, and Dujella [7] proved that the positive integer d with the above property is unique.

The purpose of this paper is to characterize linear binary recursive sequences which possess the similar property as the above property of Fibonacci numbers.

We will consider binary recursive sequences of the form

$$G_{n+1} = AG_n - G_{n-1}, \tag{1}$$

where A, G_0, and G_1 are integers. We call the sequence (G_n) *nondegenerated* if $|G_0| + |G_1| > 0$ and the quotient of the roots $\alpha, \beta \in \mathbf{C}$ of the characteristic equation of G_n,

$$x^2 - Ax + 1 = 0,$$

is not a root of unity. Let $D = A^2 - 4$, $C = G_1^2 - AG_0G_1 + G_0^2$. Then nondegeneracy implies that $|A| \geq 3$ and $C \neq 0$. Solving recurrence (1), we obtain

$$G_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta},$$

where $a = G_1 - G_0\beta$, $b = G_1 - G_0\alpha$.

Definition 1: Let k be an integer. A sequence (G_n) is said to have the property $P(k)$ if both $G_nG_{n+1} + k$ and $G_nG_{n+2} + k$ are perfect squares for all $n \geq 0$.

With this notation, we may say that the sequence (F_{2n}) has the property $P(1)$.

Our main result is the following theorem.

Theorem 1: Let (G_n) be a nondegenerated binary recursive sequence given by (1). If G_n has the property $P(k)$ for some $k \in \mathbf{Z}$, then $A = 3$ and $k = G_0^2 - 3G_0G_1 + G_1^2$.

Remark 1: The sequences from Theorem 1 have the form

$$G_n = G_1 F_{2n} - G_0 F_{2n-2},$$

and for $G_0 = 0$ and $G_1 = 1$ we obtain exactly the sequence (F_{2n}). Note that the converse of Theorem 1 is also valid. This follows from the formula (F_{2n}) proved below, and the general fact that if $ab + k = r^2$ then $a(a+b-2r)+k = (a-r)^2$.

2. PROOF OF THEOREM 1

Assume that k is an integer such that the sequence (G_n) has the property $P(k)$. This implies that $G_n G_{n+2} + k$ is a perfect square for all $n \geq 0$. On the other hand,

$$G_n G_{n+2} = \frac{a^2\alpha^{2n+2} + b^2\beta^{2n+2} - ab(\alpha\beta)^n(\alpha^2+\beta^2)}{(\alpha-\beta)^2}$$

$$= \left(\frac{a\alpha^{n+1} - b\beta^{n+1}}{\alpha-\beta}\right)^2 - \frac{ab(\alpha\beta)^n(\alpha-\beta)^2}{(\alpha-\beta)^2}$$

$$= G_{n+1}^2 - ab = G_{n+1}^2 - C.$$

Hence, $G_{n+1}^2 - C + k$ is a perfect square for all $n \geq 0$. This implies that $k = C$.

Our problem is now reduced to find sequences such that $G_n G_{n+1} + C$ is a perfect square for all $n \geq 0$.

We have $G_{n+1}^2 - AG_nG_{n+1} + G_n^2 = C$ (see [9]). Denote $G_nG_{n+1} + C = G_n^2 - (A-1)G_nG_{n+1} + G_{n+1}^2$ by H_n. It can be verified easily that the sequence (H_n) satisfies the recurrence relation

$$H_{n+1} = (A^2-2)H_n - H_{n-1} - C(A^2 - A - 4).$$

Finally, put $S_n = (A^2-4)H_n - C(A^2-A-4)$. Then the sequence (S_n) satisfies the homogeneous recurrence relation

$$S_{n+1} = (A^2-2)S_n - S_{n-1}.$$

Denote the polynomial $(A^2-4)x^2 - C(A^2-A-4)$ by $R(x)$. Then our condition implies that, for every $n \geq 0$, there exist $x \in \mathbf{Z}$ such that

$$S_n = R(x). \tag{2}$$

Therefore, equation (2) has infinitely many solutions.

Let $D_1 = (A^2-2)^2 - 4 = A^2(A^2-4)$ and $C_1 = S_1^2 - S_0S_2 = -(A^2-4)A^2C^2$ be the discriminant and the characteristic of the sequence (S_n), respectively. Assume also that

$$S_n = \frac{a_1\alpha^{2n} - b_1\beta^{2n}}{\alpha^2-\beta^2} \quad \text{for some } a_1 \text{ and } b_1,$$

and put

$$T_n = a_1\alpha^{2n} + b_1\beta^{2n} \quad \text{for all } n \geq 0.$$

Then, since

$$T_n^2 = D_1S_n^2 + 4C_1 \quad \text{for all } n \geq 0,$$

and since the equation $S_n = R(x)$ has infinitely many integer solutions (n, x), it follows that the equation

$$y^2 = D_1R(x)^2 + 4C_1$$

has infinitely many integer solutions (x, y). By a well-known theorem of Siegel [20], we get that the polynomial $F(X) = D_1R(X)^2 + 4C_1$ has at most two simple roots. Since F is of degree 4, it follows that F must have a double root. Notice that

$$F'(X) = 2D_1R(X)R'(X) = 4(A^2-4)D_1R(X)X.$$

Certainly, F and R cannot have a common root because this would imply that $C_1 = 0$, which is impossible since (G_n) is nondegenerated. Hence, $F(0) = 0$, which is equivalent to

$$A^2(A^2-4)[C(A^2-A-4)]^2 - 4A^2(A^2-4)C^2 = 0. \tag{3}$$

Formula (3) implies that $A^2 - A - 4 = \pm 2$.

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If $A^2 - A - 4 = 2$, then $A = 3$ or $A = -2$, and if $A^2 - A - 4 = -2$, then $A = 2$ or $A = -1$. Since we assumed that the sequence (G_n) is nondegenerated, i.e., $|A| \geq 3$, we conclude that $A = 3$. □

Remark 2: In degenerate cases with $A = 0$, ± 1, ± 2, the sequence (G_n) also may have property $P(k)$ for some $k \in \mathbb{Z}$. For example, for $A = 2$, the sequence $G_n = a$ has property $P(b^2 - a^2)$; for $A = 0$, the sequence $G_{2n} = 0$, $G_{4n+1} = 2ab$, $G_{4n+3} = -2ab$ has property $P((a^2+b^2)^2)$; for $A = -1$, the sequence $G_{3n} = a$, $G_{3n+1} = b$, $G_{3n+2} = -a-b$ has property $P(a^2+ab+b^2)$. Here, a and b are arbitrary integers.

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REFERENCES

1. J. Arkin, D. C. Arney, F. R. Giordano, R. A. Kolb, & G. E. Bergum. "An Extension of an Old Classical Diophantine Problem." *Applications of Fibonacci Numbers* **5**:45-48. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
2. G. Berzsenyi. "Problems, Puzzles, Paradoxes." *Consortium* **25** (1988):5.
3. M. N. Deshpande & G. E. Bergum. "Interesting Arrays Associated with Fibonacci Sequences." *Applications of Fibonacci Numbers* **6**:85-92. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.
4. A. Dujella. "Generalization of a Problem of Diophantus." *Acta Arith.* **65** (1993):15-27.
5. A. Dujella. "Generalized Fibonacci Numbers and the Problem of Diophantus." *The Fibonacci Quarterly* **34.2** (1996):164-75.
6. A. Dujella. "A Problem of Diophantus and Pell Numbers." *Applications of Fibonacci Numbers* **7**:61-68. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.
7. A. Dujella. "A Proof of the Hoggatt-Bergum Conjecture." *Proc. Amer. Math. Soc.* **127** (1999):1999-2005.

8. V. E. Hoggatt, Jr., & G. E. Bergum. "A Problem of Fermat and the Fibonacci Sequence." *The Fibonacci Quarterly* **15.4** (1977):323-30.
9. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* **32** (1965):437-46.
10. A. F. Horadam. "Generalization of a Result of Morgado." *Portugaliae Math.* **44** (1987): 131-36.
11. B. W. Jones. "A Second Variation on a Problem of Diophantus and Davenport." *The Fibonacci Quarterly* **16.2** (1978):155-65.
12. C. Long & G. E. Bergum. "On a Problem of Diophantus." *Applications of Fibonacci Numbers* **2:**183-91. Ed. A. N. Philippou, A. F. Horadam, & G. E. Bergum. Dordrecht: Kluwer, 1988.
13. S. P. Mohanty & A. M. S. Ramasamy. "On $P_{r,k}$ Sequences." *The Fibonacci Quarterly* **23** (1985):36-44.
14. J. Morgado. "Generalization of a Result of Hoggatt and Bergum on Fibonacci Numbers." *Portugaliae Math.* **42** (1983-1984):441-45.
15. J. Morgado. "Note on a Shannon Theorem Concerning the Fibonacci Numbers and Diophantine Quadruples." *Portugaliae Math.* **48** (1991):429-39.
16. J. Morgado. "Note on the Chebyshev Polynomials and Applications to the Fibonacci Numbers." *Portugaliae Math.* **52** (1995):363-78.
17. I. Nemes & A. Pethö. "Polynomial Values in Linear Recurrences II." *J. Number Theory* **24** (1986):47-53.
18. A. M. S. Ramasamy. "A Remarkable Sequence." *Banyan Math. J.* **2** (1995):69-76.
19. A. G. Shannon. "Fibonacci Numbers and Diophantine Quadruples: Generalizations of Results of Morgado and Horadam." *Portugaliae Math.* **45** (1988):165-69.
20. C. L. Siegel. "The Integer Solutions of the Equation $y^2 = ax^n + bx^{n-1} + \cdots + k$." *J. London Math. Soc.* **1** (1926):66-68.
21. Gh. Udrea. "A Note on the Sequence $(W_n)_{n\geq 0}$ of A. F. Horadam." *Portugaliae Math.* **53** (1996):143-55.

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