MATRIX POWERS OF COLUMN-JUSTIFIED PASCAL TRIANGLES AND FIBONACCI SEQUENCES

Rhodes Peele and Pantelimon Stanica
Department of Mathematics, Auburn University Montgomery, Montgomery, AL 36124
e-mail: peelhor@mindspring.com; stanpan@strudel.aum.edu
(Submitted January 2000-Final Revision October 2000)

1. MOTIVATION

It is known that if L_n, respectively R_n, are $n \times n$ matrices with the $(i, j)^{\text{th}}$ entry the binomial coefficient $\binom{i-1}{j-1}$, respectively $\binom{i-1}{n-j}$, then $L_n^2 \equiv I_n \pmod 2$, respectively $R_n^3 \equiv I_n \pmod 2$, where I_n is the identity matrix of dimension $n > 1$ (see, e.g., Problem P10735 in the May 1999 issue of *Amer. Math. Monthly*).

The entries of L_n form a left-justified Pascal triangle and the entries of R_n result from taking the mirror-image of this triangle with respect to its first column.

The questions we ask are: Can this result be extended to other primes or, better yet, is it possible to find a closed form for the entries of powers of L_n and R_n?

L_n succumbs easily, as we shall see in our first result. R_n in turn fights back, since closed forms for its powers are not found. However, we show a beautiful connection between matrices similar to R_n and the Fibonacci numbers. If $n = 2$, the connection is easily seen, since

$$R_2^e = \begin{pmatrix} F_{e-1} & F_e \\ F_e & F_{e+1} \end{pmatrix}.$$

A simple consequence of our results is that the order of L_n modulo a prime p is p, and the order of R_n modulo p divides four times the entry point of the Fibonacci sequence modulo p.

2. HIGHER POWERS OF L_n AND R_n

The first approach that comes to mind is to find a closed form for all entries of powers of L_n and R_n. It is not difficult to obtain all the powers of L_n. Denoting the entries of the e^{th} power of L_n by $l_{i,j}^{(e)}$, we can prove

Theorem 1: The entries of L_n^e are

$$l_{i,j}^{(e)} = e^{i-j}\binom{i-1}{j-1}. \tag{1}$$

Proof: We use induction on e. The result is certainly true for $e = 1$. Now, using induction and matrix multiplication,

$$l_{i,j}^{(e+1)} = \sum_{s=1}^{n}\binom{i-1}{s-1}e^{s-j}\binom{s-1}{j-1} \overset{[5],\text{p.}3}{=} \sum_{s=1}^{n} e^{s-j}\binom{i-1}{j-1}\binom{i-j}{i-s}$$
$$= \binom{i-1}{j-1}\sum_{k=0}^{i-j} e^{i-j-k}\binom{i-j}{k} = \binom{i-1}{j-1}(e+1)^{i-j}. \qquad \square$$

To prove a similar result for R_n is no easy matter. In fact, except for a few lower-dimensional cases and a few of its rows/columns, simple closed forms for the entries of R_n^e are not found.

In the sequel, we consider the tableau with entries a_{ij}, $i \geq 1$, $j \geq 0$, satisfying

$$a_{i,\,j-1} = a_{i-1,\,j-1} + a_{i-1,\,j}, \tag{2}$$

with boundary conditions $a_{1,\,n} = 1$, $a_{1,\,j} = 0$, $j \neq n$. We shall use the following consequences of the boundary conditions and recurrence (2): $a_{i,\,j} = 0$ for $i + j \leq n$, and $a_{i,\,n+1} = 0$, $1 \leq i \leq n$ [in fact, we use *only* these consequences and (2)]. The matrix R will be defined as $(a_{i,\,j})_{i=1\ldots n,\,j=1\ldots n}$. We treat the second and third powers first, since it gives us the idea about the general case. To clear up the mysteries of some of the steps in our calculations, we will refer to matrix multiplication as *m.m.* and the boundary conditions as *b.c.*.

Lemma 2: The entries of the matrix R^2 satisfy

$$b_{i,\,j+1} = b_{i-1,\,j+1} + 2b_{i-1,\,j} - b_{ij},\ 2 \leq i \leq n,\ 1 \leq j \leq n-1, \tag{3}$$

and the entries of R^3 satisfy

$$c_{i+1,\,j} = 2c_{i,\,j} + 3c_{i,\,j-1} - 2c_{i+1,\,j-1},\ 1 \leq i \leq n-1,\ 2 \leq j \leq n. \tag{4}$$

Proof: Using matrix multiplication and (2), we obtain

$$\begin{aligned} b_{i,\,j+1} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{i,\,s}a_{s,\,j+1} \overset{(2)}{=} \sum_{s=1}^{n} a_{i,\,s}(a_{s+1,\,j} - a_{s,\,j}) \\ &= \sum_{s=1}^{n} a_{i,\,s}a_{s+1,\,j} - \sum_{s=1}^{n} a_{i,\,s}a_{s,\,j} \overset{m.m.}{=} \sum_{s=1}^{n} a_{i,\,s}a_{s+1,\,j} - b_{i,\,j}. \end{aligned} \tag{5}$$

Therefore, denoting $S_{i,\,j} = \sum_{s=1}^{n} a_{i,\,s}a_{s+1,\,j}$, we obtain

$$b_{i,\,j+1} + b_{i,\,j} = S_{i,\,j}. \tag{6}$$

If $2 \leq i \leq n$ and $1 \leq j \leq n$,

$$\begin{aligned} S_{i,\,j} &= \sum_{s=1}^{n} (a_{i-1,\,s} + a_{i-1,\,s+1})a_{s+1,\,j} \overset{t=s+1}{=} S_{i-1,\,j} + \sum_{t=2}^{n+1} a_{i-1,\,t}a_{t,\,j} \\ &\overset{m.m.}{=} S_{i-1,\,j} + b_{i-1,\,j} + a_{i-1,\,n+1}a_{n+1,\,j} - a_{i-1,\,1}a_{1,\,j} \overset{b.c.}{=} S_{i-1,\,j} + b_{i-1,\,j}. \end{aligned}$$

Using (6) in the previous recurrence, we obtain $b_{i,\,j+1} + b_{i,\,j} = b_{i-1,\,j+1} + b_{i-1,\,j} + b_{i-1,\,j}$, which gives us (3).

If the relations (3) are satisfied, we obtain, for $j \geq 2$,

$$\begin{aligned} c_{i,\,j} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{i,\,s}b_{s,\,j} \overset{(3)}{=} \sum_{s=1}^{n} a_{i,\,s}(b_{s-1,\,j} + 2b_{s-1,\,j-1} - b_{s,\,j-1}) \\ &= \sum_{s=1}^{n} a_{i,\,s}b_{s-1,\,j} + 2\sum_{s=1}^{n} a_{i,\,s}b_{s-1,\,j-1} - c_{i,\,j-1} = T_{i,\,j} + 2T_{i,\,j-1} - c_{i,\,j-1}, \end{aligned}$$

where $T_{i,\,j} = \sum_{s=1}^{n} a_{i,\,s}b_{s-1,\,j}$. Furthermore, for $i \leq n-1$,

$$\begin{aligned} T_{i,\,j} &\overset{(2)}{=} \sum_{s=1}^{n} (a_{i+1,\,s-1} - a_{i,\,s-1})b_{s-1,\,j} \overset{m.m.}{=} c_{i+1,\,j} + a_{i+1,\,0}b_{0,\,j} - a_{i+1,\,n}b_{n,\,j} - c_{i,\,j} - a_{i,\,0}b_{0,\,j} + a_{i,\,n}b_{n,\,j} \\ &\overset{(2)}{=} c_{i+1,\,j} - c_{i,\,j} + a_{i,\,1}b_{0,\,j} - a_{i,\,n+1}b_{n,\,j} \overset{b.c.}{=} c_{i+1,\,j} - c_{i,\,j}. \end{aligned}$$

Therefore,

$$c_{i,j} = T_{i,j} + 2T_{i,j-1} - c_{i,j-1} = c_{i+1,j} - c_{i,j} + 2c_{i+1,j-1} - 2c_{i,j-1} - c_{i,j-1},$$

which will produce the equations (4). □

Corollary 3: The entries of the second and third power of R can be expressed in terms of the entries of the previous row:

$$b_{i+1,j} = b_{i,j} - \sum_{k=1}^{j-1}(-1)^k b_{i,j-k}, \quad c_{i+1,j} = 2c_{i,j} + \sum_{k=1}^{j-1}(-1)^k 2^{k-1} c_{i,j-k}.$$

We have wondered if relations similar to (3) or (4) are true for higher powers of R. It turns out that

Theorem 4: The entries $a_{i,j}^{(e)}$ of the e^{th} power of R satisfy the relation

$$F_{e-1} a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)},$$

where F_e is the Fibonacci sequence.

Proof: We show first that the entries of R^e satisfy a relation of the form

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)} \tag{7}$$

and then will proceed to find these coefficients. From Lemma 2, we observe that $\delta_1 = 0$, $\alpha_1 = 1$, $\beta_1 = 1$, $\gamma_1 = -1$, $\delta_2 = 1$, $\alpha_2 = 1$, $\beta_2 = 2$, $\gamma_2 = -1$, and $\delta_3 = 1$, $\alpha_3 = 2$, $\beta_3 = 3$, $\gamma_3 = -2$. Now, the coefficients of R^e satisfy, for $i, j \geq 2$,

$$\begin{aligned} \delta_{e-1} a_{i-1,j}^{(e)} &\overset{m.m.}{=} \sum_{s=1}^{n} \delta_{e-1} a_{i-1,s} a_{s,j}^{(e-1)} \overset{(7)}{=} \sum_{s=1}^{n} a_{i-1,s}\left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} + \gamma_{e-1} a_{s,j-1}^{(e-1)}\right) \\ &\overset{m.m.}{=} \alpha_{e-1} U_{i-1,j} + \beta_{e-1} U_{i-1,j-1} + \gamma_{e-1} a_{i-1,j-1}^{(e)}, \end{aligned} \tag{8}$$

where $U_{i,j} = \sum_{s=1}^{n} a_{i,s} a_{s-1,j}^{(e-1)}$. We evaluate, for $2 \leq i \leq n$,

$$\begin{aligned} U_{i-1,j} &\overset{(2)}{=} \sum_{s=1}^{n} (a_{i,s-1} - a_{i-1,s-1}) a_{s-1,j}^{(e-1)} \\ &\overset{m.m.}{=} a_{i,j}^{(e)} - a_{i-1,j}^{(e)} + a_{i,0} a_{0,j}^{(e-1)} - a_{i-1,0} a_{0,j}^{(e-1)} - a_{i,n} a_{n,j}^{(e-1)} + a_{i-1,n} a_{n,j}^{(e-1)} \\ &\overset{(2)}{=} a_{i,j}^{(e)} - a_{i-1,j}^{(e)} + a_{i-1,1} a_{0,j}^{(e-1)} - a_{i-1,n+1} a_{n,j}^{(e-1)} = a_{i,j}^{(e)} - a_{i-1,j}^{(e)}, \end{aligned}$$

since $a_{i-1,1} = 0$, $i \leq n$, and $a_{i-1,n+1} = 0$. Thus,

$$\alpha_{e-1} a_{i,j}^{(e)} = (\delta_{e-1} + \alpha_{e-1}) a_{i-1,j}^{(e)} + (\beta_{e-1} - \gamma_{e-1}) a_{i-1,j-1}^{(e)} - \beta_{e-1} a_{i,j-1}^{(e)}.$$

Therefore, we obtain the following system of sequences:

$$\begin{aligned} \delta_e &= \alpha_{e-1}, \\ \alpha_e &= \alpha_{e-1} + \delta_{e-1}, \\ \beta_e &= \beta_{e-1} - \gamma_{e-1}, \\ \gamma_e &= -\beta_{e-1}. \end{aligned} \tag{9}$$

From this, we deduce $\delta_e = F_{e-1}$, $\alpha_e = F_e$, $\beta_e = F_{e+1}$, $\gamma_e = -F_e$, where F_e is the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$. □

Corollary 3 can be generalized, with a little more work and anticipating (10), to obtain the elements in the $(i+1)^{\text{th}}$ row of R^e, in terms of the elements in the previous row.

Proposition 5: We have

$$F_{e-1}a_{i+1,j}^{(e)} = F_e a_{i,j}^{(e)} - \sum_{k=1}^{j-1}(-1)^{k+e}\frac{F_e^{k-1}}{F_{e-1}^k}a_{i,j-k}^{(e)}.$$

3. HIGHER POWERS OF L_n AND R_n MODULO A PRIME p

As before let L_n, respectively R_n, be defined as the matrices with entries $\binom{i-1}{j-1}$, respectively $\binom{i-1}{n-j}$. We use the notation "*matrix* $\equiv a \pmod p$" with the meaning "*matrix* $\equiv aI_n \pmod p$".

We ask the question of whether or not the order of L_n and R_n modulo a prime p is finite. We can easily prove a result for L_n using Theorem 1.

Theorem 6: The order of L_n $(n \geq 2)$ modulo p is p.

Proof: We have shown that the entries of L_n^e are $l_{i,j}^{(e)} = e^{i-j}\binom{i-1}{j-1}$ for any integer e. Thus, the entries on the principal diagonal of L_n^e are all 1. If $i \neq j$, then $p \mid l_{i,j}^{(p)}$. Assume there is an integer e with $0 < e < p$ such that $p \mid l_{i,j}^{(e)}$ for all $i \neq j$. Take $i = 2$ and $j = 1$. Then $p \mid e$ is a contradiction. Therefore, the integer p is the least integer $e > 0$ for which $p \mid l_{i,j}^{(e)}$ for all $i \neq j$, which proves our assertion. □

We can prove the finiteness of the order of R_n modulo p in a simple manner. By the Pigeonhole Principle, there exist $s < t$ such that $R_n^s \equiv R_n^t \pmod p$. Since R_n is an invertible matrix (det $R_n = (-1)^n \not\equiv 0 \pmod p$), $R_n^{t-s} \equiv I_n \pmod p$. More precise results will be proved next. In order to do that, we need some known facts about the period of the Fibonacci sequence. It was shown that the period of the Fibonacci sequence modulo m (not necessarily prime) is less than or equal to $6m$ (with equality holding for infinitely many values of m) (see P. Freyd, Problem E 3410, *Amer. Math. Monthly*, December 1990, with a solution provided in *ibid.*, March 1992). In the case of a prime, the result can be strengthened (see Theorem 7). The least integer $n \neq 0$ with the property $m \mid F_n$ is called the *entry point* modulo m.

In [1] and [7], the authors obtain (see also [6], Chs. VI-VII, for a more updated source)

Theorem 7 (Bloom-Wall): Denote the period of the Fibonacci sequence modulo p by $\mathcal{P}(p)$. Let p be an odd prime with $p \neq 5$. If $p \equiv \pm 1 \pmod 5$, then the period $\mathcal{P}(p) \mid (p-1)$. If $p \equiv \pm 3 \pmod 5$, then the entry point $e \mid (p+1)$ and the period $\mathcal{P}(p) \mid 2(p+1)$.

Remark 8: For $p = 2$, the entry point is 3 and the period is 3. In the case $p = 5$, the entry point is 5 and the period is 20.

Theorem 9: If e is the entry point modulo p of F_e, then $R_{2k}^e \equiv (-1)^{(k+1)e}F_{e-1}I_{2k} \pmod p$ and $R_{2k+1}^e \equiv (-1)^{ke}I_{2k+1} \pmod p$. Moreover, $R_n^{4e} \equiv I_n \pmod p$.

Proof: We prove by induction on e that the elements in the first row and first column of R_n^e are

$$a_{1,j}^{(e)} = \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1} \quad \text{and} \quad a_{i,1}^{(e)} = F_{e-1}^{n-i} F_e^{i-1}. \tag{10}$$

First, we deal with the elements in the first row. The first equation is certainly true for $e = 1$, if we define $0^0 = 1$. Now,

$$\begin{aligned} a_{1,j}^{(e+1)} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{1,s}^{(e)} a_{s,j} = \sum_{s=1}^{n} F_{e-1}^{n-s} F_e^{s-1} \binom{n-1}{s-1}\binom{s-1}{n-j} \\ &= \sum_{s=1}^{n} F_{e-1}^{n-s} F_e^{s-1} \binom{n-1}{j-1}\binom{j-1}{n-s} = F_e^{n-1}\binom{n-1}{j-1}\sum_{s=1}^{n}\left(\frac{F_{e-1}}{F_e}\right)^{n-s}\binom{j-1}{n-s} \\ &= F_e^{n-1}\binom{n-1}{j-1}\left(1+\frac{F_{e-1}}{F_e}\right)^{j-1} = F_e^{n-j} F_{e+1}^{j-1}\binom{n-1}{j-1}. \end{aligned}$$

Again by induction, we prove the result for the elements in the first column. The case $e = 1$ can be checked easily. Then

$$\begin{aligned} a_{i,1}^{(e+1)} &\overset{m.m.}{=} \sum_{s=1}^{n} a_{i,s} a_{s,1}^{(e)} = \sum_{s=1}^{n} \binom{i-1}{n-s} F_{e-1}^{n-s} F_e^{s-1} \\ &= F_e^{n-1} \sum_{s=1}^{n}\left(\frac{F_{e-1}}{F_e}\right)^{n-s}\binom{i-1}{n-s} = F_e^{n-1}\left(1+\frac{F_{e-1}}{F_e}\right)^{i-1} = F_e^{n-i} F_{e+1}^{i-1}. \end{aligned}$$

Let e be the entry point modulo p of the Fibonacci sequence. By Bloom-Wall's result, we have $e \le p+1$. Using Theorem 4, we obtain $F_{e-1} a_{i,j}^{(e)} \equiv F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)} \pmod p$. Thus,

$$F_{e-1} a_{i,j}^{(e)} \equiv F_{e+1} a_{i-1,j-1}^{(e)} \pmod p. \tag{11}$$

Since $F_{e-1} + F_e = F_{e+1}$, $p \mid F_e$, and $p \nmid F_{e-1}$, we obtain $F_{e-1} \equiv F_{e+1} \pmod p$ and

$$a_{i,j}^{(e)} \equiv a_{i-1,j-1}^{(e)} \pmod p. \tag{12}$$

We see from what was proved above that, modulo p, the elements in the first row and column of $R_n \pmod p$ are all zero, except for the one in the first position, which is $F_{e-1}^{n-1} \not\equiv 0 \pmod p$. Using (12), we get $R_n^e \equiv F_{e-1}^{n-1} I_n \pmod p$. Using Cassini's identity $F_{e-1}F_{e+1} - F_e^2 = (-1)^e$ (see [2], p. 292), we obtain $F_{e-1}^2 \equiv F_{e+1}^2 \equiv (-1)^e \pmod p$. If $n = 2k$, then

$$F_{e-1}^{n-1} = F_{e-1}^{2k-1} \equiv (F_{e-1}^2)^k F_{e-1}^{-1} \equiv (-1)^{ke} F_{e-1}^{-1} \equiv (-1)^{(k+1)e} F_{e-1} \pmod p.$$

If $n = 2k+1$, then $F_{e-1}^{n-1} = F_{e-1}^{2k} \equiv (F_{e-1}^2)^k \equiv (-1)^{ke} \pmod p$.

The previous two congruences replaced in $R_n^e \equiv F_{e-1}^{n-1} I_n \pmod p$, will give the first two assertions of our theorem.

It is well known (a very particular case of Matijasevich's lemma) that $F_{2e-1} = F_{e-1}^2 + F_e^2 \equiv F_{e-1}^2 \pmod{F_e^2}$, so $F_{2e-1}^2 \equiv 1 \pmod p$. Thus, since F_m divides F_{sm} for all m and s (in particular, for $s = 2$, $m = e$), it follows that $F_{2e} \equiv 0 \pmod p$ and $R_n^{4e} = (R_n^{2e})^2 \equiv (F_{2e-1}^2)^{n-1} I_n \equiv I_n \pmod p$. □

Remark 10: We remark here the fact that the bound $4e$ for the order of R is tight. That can be seen by taking, for example, the prime 13, since the entry point for the Fibonacci sequence is 7, and the order of R_{4k} is 28.

Using some elementary number theory, we can prove

Theorem 11: If $p \mid F_{p-1}$, then $R_n^{p-1} \equiv I_n \pmod{p}$.

Proof: We observe that, since $p \mid F_{p-1}$, we have $p \equiv \pm 1 \pmod{5}$, otherwise, $p \equiv \pm 2 \pmod{5}$, and by Bloom-Wall's theorem, the entry point e divides $p+1$. Thus, $e | p-1$ and $e | p+1$. Therefore, e must be 2. This is not possible because $F_2 = 1$, which is not divisible by any prime. So, $p \equiv \pm 1 \pmod{5}$ and $F_p \equiv F_{p-2} \pmod{p}$. Thus, $R_n^{p-1} \equiv F_{p-2}^{n-1} I_n \equiv F_p^{n-1} I_n \pmod{p}$.

By the previous Bloom-Wall theorem, $\mathcal{P}(p) \mid (p-1)$; therefore, $F_{p-1} \equiv 0$, $F_p \equiv 1$, $F_{p+1} \equiv 1$, etc. Hence, $R_n^{p-1} \equiv F_p^{n-1} I_n \equiv I_n \pmod{p}$. □

Another interesting result is the following theorem.

Theorem 12: If $p | F_{p+1}$, then $R_{2k+1}^{p+1} \equiv I_{2k+1} \pmod{p}$ and $R_{2k}^{p+1} \equiv -I_{2k} \pmod{p}$.

Proof: Assume $p = 2$. The entry point of the Fibonacci sequence modulo 2 is $e = 3$. Since $F_2 = 1$, Theorem 9 shows the result in this case. Assume $p > 2$. We know that in this case we must have $p \equiv \pm 2 \pmod{5}$. Using the known formula (see, e.g., [3], Theorem 180)

$$F_j = 2^{1-j}\left[\binom{j}{1} + 5\binom{j}{3} + 5^2\binom{j}{5} + \cdots\right],$$

taking $j = p$, and using Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$, we obtain

$$F_p \equiv 5^{(p-1)/2}\binom{p}{p} \equiv -1 \pmod{p},$$

since, for the primes $\equiv \pm 2 \pmod{5}$, 5 is a quadratic nonresidue.

When n is odd, $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (F_p^2)^{\frac{n-1}{2}} I_n \equiv I_n \pmod{p}$. Consider the case of n even. Since $F_p \equiv -1 \pmod{p}$, we have $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (-1)^{n-1} I_n \equiv -I_n \pmod{p}$. □

The proofs of the previous two theorems imply

Corollary 13: If $p \equiv \pm 1 \pmod{5}$ and $p-1$ is the entry point for the Fibonacci sequence modulo p, then the period is exactly $p-1$. If $p \equiv \pm 2 \pmod{5}$ and $p+1$ is the entry point for the Fibonacci sequence modulo p, then the period is exactly $2(p+1)$.

Corollary 14: The order of $R_n \pmod{p}$ is less than or equal to $2(p+1)$ and the bound is met.

Proof: If $p \equiv \pm 1 \pmod{5}$, then the order of $R_n \pmod{p}$ is $\le p-1$. If $p \equiv \pm 2 \pmod{5}$, then $F_p \equiv -1 \pmod{p}$. Therefore, $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (-1)^{n-1} I_n \pmod{p}$. Thus, $R_n^{2p+2} \equiv I_n \pmod{p}$. The bound is met for all primes $p \equiv \pm 2 \pmod{5}$ and all even integers n. □

4. FURTHER PROBLEMS AND RESULTS

The inverses of R_n and L_n are not difficult to find. We have

Theorem 15: The inverse of

$$L_n = \left(\binom{i-1}{j-1}\right)_{1 \le i, j \le n} \quad \text{is} \quad L_n^{-1} = \left((-1)^{i+j}\binom{i-1}{j-1}\right)_{1 \le i, j \le n}.$$

The inverse of

$$R_n = \left(\binom{i-1}{n-j} \right)_{1 \le i,\, j \le n} \text{ is } R_n^{-1} = \left((-1)^{n+i+j+1} \binom{n-i}{j-1} \right)_{1 \le i,\, j \le n}.$$

Proof: We have

$$\sum_s^n (-1)^{i+s} \binom{i-1}{s-1} \binom{s-1}{j-1} \overset{[5], \text{p.3}}{=} \sum_s^n (-1)^{i+s} \binom{i-1}{j-1} \binom{i-j}{i-s} \overset{k=i-s}{=} \binom{i-1}{j-1} \sum_{k=0}^{i-j} (-1)^k \binom{i-j}{k},$$

which is 0, unless $i = j$, in which case it is 1. A similar analysis for R_n will produce its inverse. □

Another approach to find a closed form for all entries of powers of R_n would be to find all eigenvalues of R_n, and use the diagonalization of the matrix to find the entries of R_n. We found the following empirically and we state it as a conjecture.

Conjecture 16: Denote $\phi = \frac{1+\sqrt{5}}{2}$, $\overline{\phi} = \frac{1-\sqrt{5}}{2}$. The eigenvalues of R_n are:

(a) $\{(-1)^{k+i}\phi^{2i-1}, (-1)^{k+i}\overline{\phi}^{2i-1}\}_{i=1,\dots,k}$ if $n = 2k$.

(b) $\{(-1)^k\} \cup \{(-1)^{k+i}\phi^{2i}, (-1)^{k+i}\overline{\phi}^{2i}\}_{i=1,\dots,k}$ if $n = 2k+1$.

Another venue of research would be to study the matrices associated to other interesting sequences—Lucas, Pell, etc.—and we will approach this matter elsewhere.

Note Added to Proof: Recently, the above-mentioned conjecture was settled in the affirmative, independently, by P. Stanica and R. Peele, by D. Callan, and by H. Prodinger.

ACKNOWLEDGMENT

The authors would like to thank the anonymous referee for his helpful comments, which improved significantly the presentation of the paper.

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AMS Classification Numbers: 05A10, 11B39, 11B65, 11C20, 15A33

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